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Turbulence in the Gromov space

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Abstract

This paper extends the theory of turbulence of Hjorth to certain classes of equivalence relations that cannot be induced by Polish actions. It applies this theory to analyze the quasi-isometry relation and finite Gromov-Hausdorff distance relation in the space of isometry classes of pointed proper metric spaces, called the Gromov space.

Contents

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1 Introduction

This article originates in the study of the generic geometry of the leaves of a foliated space. Those studies aim at answering the following question: what geometric properties are common to all (or to almost all, either in category theoretical sense or in a measure theoretic sense) the leaves? Examples of such geometric properties include: (a) number of ends; (b) growth type; (c) continuous spectrum; (d) asymptotic dimension; (e) coarse cohomology.

Our original approach to studying this question was as follows. Gromov [4, Section 6], [3] described a space, the Gromov space of the title which is denoted here by $\mathcal{M}_*$, whose points are isometry classes of pointed complete proper metric spaces. It is endowed with a topology which resembles the compact open topology on the space of continuous functions on $\mathbf{R}$. A foliated space, X, endowed with a metric on the leaves under which each leaf is a complete Riemannian manifold admits a canonical continuous mapping into the Gromov space $\mathcal{M}_*$. This mapping assigns to a point x in X the isometry class of the pointed metric space (F_x, x), where F_x is the holonomy covering (based at x) of the leaf of X that contains the point x.

The space $\mathcal{M}_*$ supports several equivalence relations of geometric interest. For example, the relation of being quasi-isometric, the relation of being at finite Gromov-Hausdorff distance, the relation of being bi-Lipschitz equivalent, and others. Obviously the canonical mapping of X into $\mathcal{M}_*$ is invariant with respect to the equivalence relation "being in the same leaf" over X and any of the equivalence relations mentioned above over $\mathcal{M}_*$.

Somewhat informally, a geometric property can be thought of as a mapping, $\gamma : \mathcal{M}_* \to P$, of $\mathcal{M}_*$ into a space P that is constant on the equivalence classes of one of the equivalence relations over $\mathcal{M}_*$ mentioned above. The general question posed in the first paragraph is thus: what type of situations will make the mapping $X \to P$ given as the composite of γ with the canonical mapping of X into $\mathcal{M}_*$ be constant on a large saturated subset of X? Fairly standard arguments of topological dynamics prove that if X is topologically ergodic (i.e. transitive, that is, has a dense leaf) and γ and P have suitable topological properties, then the geometric invariant must be constant on a residual saturated subset of X.

More generally, P may be endowed with an equivalence relation having suitable topological properties and γ be invariant with respect to the geometric equivalence relation over $\mathcal{M}_*$ being studied and that equivalence relation over P. Then the opening question is formulated thus: Is there a residual saturated subset of X over which γ is constant up to equivalence in P? This property is precisely formulated below and is called generic ergodicity with respect to the relation over P.

A section by section description of the contents of this paper now follows. In Section 2 we analyze a topology on the space of subsets of a space appropriate for working with equivalence relations. This topology is essentially the Vietoris topology [10] but the properties that we need are not found on the literature on the topic. These topological properties are of a categorical nature, and are needed to obtain a new version (Theorem 2.17) of the Kuratowski-Ulam theorem [9, p. 222] which describes how topological properties of a subset of a space over which an equivalence relation is defined translate to properties of the intersection of that set with the orbits of the

equivalence relation (indeed, our version of the Kuratowski-Ulam theorem also applies to non-equivalence relations). The Kuratowski-Ulam theorem is one of key tools for studying generic ergodicity of one relation with respect to another.

In Section 3 we briefly review the basic concepts of classification of equivalence relations. Complexity of an equivalence relation is quantified by comparing that relation with one of the standard examples, like the identity relation over a space or the relation "being on the same orbit" of a group action, for instance. Two concepts used for describing the relative complexity of two equivalence relations, E over X and F over Y, are reducibility and generic ergodicity. The relation E is Borel reducible to F if there is an (E, F)-invariant Borel mapping $\theta : X \to Y$ (that is, θ takes equivalence classes of E into equivalence classes of F) such that the mapping $\bar{\theta} : X/E \to Y/F$ induced by θ between quotient spaces is injective. The relation E is generically F-ergodic if for any (E, F)-invariant Borel mapping $\theta : X \to Y$ there is a residual saturated subset $C \subset X$ such the mapping $\bar{\theta} : C/E \to Y/F$ induced by θ between quotient spaces is constant.

The more elementary equivalence relations, called smooth or concretely classifiable, are those Borel reducible to the identity relation over a standard Borel space. For example, the equivalence relation of being isometric in the set of compact metric spaces is smooth because the space of equivalence classes of this relation is itself a Polish metric space when endowed with the Gromov-Hausdorff metric.

At a higher level of complexity are the equivalence relations that are classifiable by isomorphism classes of countable structures. A countable structure is a structure on the natural numbers that is determined by a countable family of relations. This set of countable structures is endowed with a Polish topology, and carries a continuous action of S_∞, the Polish group of permutations of the natural numbers, so that two countable structures are isomorphic if and only if they are in the same orbit of this S_∞-action. Thus, an equivalence relation over a Borel space is classifiable by countable structures if it is Borel reducible to the relation given by the action of S_∞ on the space of countable structures. A variety of examples of equivalence relations that are classifiable by countable structures and which arise in dynamical systems are given in Kechris [7], Hjorth [5, Preface].

A key concept in the analysis of the complexity of Polish group actions (classification by countable structures and generic ergodicity) is that of turbulence, introduced by Hjorth [5]. For a Polish group action to be turbulent, not only the action must be highly complex (transitive, minimal) but the group itself must be highly complex (actions of locally compact groups are not turbulent).

One of the results of the present paper is that the relation of being at finite Gromov-Hausdorff distance in the Gromov space $\mathcal{M}_*$ is not reducible to an equivalence relation given by a Polish group action. Therefore, the theory of turbulence for group actions needs to be amplified to a theory of turbulence for general equivalence relations. This amplification is carried out in this paper in the setting of uniform equivalence relations. A uniform equivalence relation is a pair, $(\mathcal{V}, E)$, consisting of a uniformity $\mathcal{V}$ with a distinguished entourage E which is an equivalence relation. A first example of uniform equivalence relation arises from the continuous action of a Polish group, G, on a Polish space, X. The uniformity on X is generated by the entourages $\{\, (x, gx) \mid x \in X \;\&\; g \in W \,\}$, where $\{W\}$ is a neighborhood system of the identity of G, and the equivalence

relation is given by xE_Gy if and only if $gx = y$ for some $g \in G$. A second example arises from a distance-like mapping $d : X \times X \to [0, \infty]$ that satisfies the standard properties of a distance but it is allowed to have $d(x, y) = \infty$ for some $x, y \in X$. The uniformity is generated by $\{ (x, x') \mid d(x, x') < \epsilon \}$, $\epsilon > 0$, and the equivalence relation is given by xE_dy if and only if $d(x, y) < \infty$. The pair (d, E_d) (or simply d) is called a metric equivalence relation.

As said, the main goal of this paper is to analyze the complexity of several metric equivalence relations in the Gromov space, proving that they are turbulent and not reducible to Polish actions. A general scheme for this kind of analysis is described in Section 6, and consists in a sequence of hypothesis that collective-wise will eventually guarantee that a metric equivalence relation that satisfies them is turbulent and is not reducible to the equivalence relation given by a Polish action.

In Section 7, as a prelude to the study of the "turbulent dynamics" of the Gromov space, we study the equivalence relation "being at finite uniform distance" over the space of continuous functions on $\mathbf{R}$.

Section 8 reviews the construction of the Gromov space $\mathcal{M}_*$, and the pointed Gromov-Hausdorff distance with possible infinite values, d_{GH}, between isometry classes of pointed proper metric spaces. This distance defines the relation "being at finite Gromov-Hausdorff distance" over $\mathcal{M}_*$, denoted by E_{GH}. Another equivalence relation over $\mathcal{M}_*$ introduced in this section is "being quasi-isometric," denoted by E_{QI}, which turns out to be induced by a distance function with possible infinite values, d_{QI}.

Section 9 analyzes the metric equivalence relation given by (d_{GH}, E_{GH}) over $\mathcal{M}_*$ and culminates in the following theorem.

Theorem 9.3. (i) *The metric equivalence relation (d_{GH}, E_{GH}) is turbulent.*

(ii) *For every Polish S_∞-space Y, the equivalence relation E_{GH} is generically $E^Y_{S_\infty}$-ergodic.*

(iii) *For every Polish group G and every Polish G-space X, the equivalence relation $E_{GH} \not\leq_B E^X_G$.*

Section 10 analyzes the metric equivalence relation (d_{QI}, E_{QI}) over $\mathcal{M}_*$ and culminates is an analogous result.

Theorem 10.5. (i) *The metric equivalence relation (d_{QI}, E_{QI}) is turbulent.*

(ii) *For every Polish S_∞-space Y, the equivalence relation E_{QI} is generically $E^Y_{S_\infty}$-ergodic.*

Parts (ii) of these results apply to the case of Y being the S_∞-space of countable structures and thus can be seen as justification of a metric space version of the so called Gromov's principle for discrete groups: "No statement about all finitely presented groups is both non-trivial and true."

Problems on the classification theory of metric spaces were brought to light by Vershik [16]. That paper revisits the Uhryson space, a universal Polish metric space (every Polish space is isometric to a closed subset of Uhryson space), and uses it to show that the classification of Polish metric spaces up to isometry is not smooth. The

problem of describing the complexity of the classification of Polish metric spaces up to isometry, and certain subfamilies of Polish metric spaces, was taken up later in Gao-Kechris [2]. Work on the classification of the quasi-isometry relation over the space of finitely generated groups was done by Thomas [15].

While the Gromov space and the Uhryson space are certainly not unrelated, they are not interchangeable for analyzing our initial problem on the generic geometry of the leaves of a foliated space. In particular, there is not such object as the canonical mapping from a foliated space into the hyperspace of closed subsets of Uhryson space.

Now that we have analyzed the dynamical structure of the Gromov space, it makes sense to revisit the initial problem at the beginning of this introduction, using our approach of studying generic geometric properties of the leaves of a foliated space via the canonical mapping of the foliated space into the Gromov space. For instance, we can formulate questions like what conditions on a foliated space guarantee that the restriction of d_{GH} or d_{QI} to its canonical image in $\mathcal{M}_*$ is turbulent. It also makes sense to analyze how several conditions on the dynamics of the foliated space affect generic geometric properties of its leaves. This is particularly dramatic for codimenson one foliated manifolds with sufficient transverse smoothness. For example, by a theorem of Duminy, an exceptional minimal set of one of such foliations must contain a leaf with a Cantor set of ends, but it is not know if it contains a residual set of leaves with a Cantor set of ends.

2 Continuous relations

Let $\mathbf{2} = \{0, 1\}$ denote the two-point set. If X is any set, then 2^X, the set of mappings $X \to \mathbf{2}$, is naturally identified with the set of all subsets of X by means of the characteristic mapping of a subset.

For a subset $A \subset X$, let

$$P_A = \{\, B \subset X \mid B \cap A \neq \emptyset \,\}\ .$$

There is a natural identification

$$2^A = 2^X \setminus P_{X \setminus A}\ . \tag{1}$$

Moreover

$$P_\emptyset = \emptyset \qquad \text{and} \qquad P_X = 2^X \setminus \{\emptyset\}\ ,$$

and for any family $\{A_i \mid i \in I\}$ of subsets of X,

$$P_{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} P_{A_i} \qquad \text{and} \qquad P_{\bigcap_{i\in I} A_i} \subset \bigcap_{i\in I} P_{A_i}\ .$$

If X is a topological space, then $\mathbf{2}^X$ becomes naturally a topological space when endowed with the topology that has the family $\{P_U \mid U \text{ open in } X\}$ as a subbase. This is called the Vietoris topology (Vietoris [17], Michael [11]). In what follows, provided that X is a topological space and unless otherwise stated, 2^X will always be endowed with the Vietoris topology.

If $\mathcal{B}$ is a base for a topology on X, then

$$\left\{ \bigcap_{U \in \mathcal{C}} P_U \mid \mathcal{C} \text{ is a finite subset of } \mathcal{B} \right\}$$

is a base for the Vietoris topology on $\mathbf{2}^X$. It follows in particular that 2^X is second countable if X is second countable.

A (binary) relation, E, over sets, X and Y, is a subset $E \subset X \times Y$. The set X is called the *source* of E and the set Y is called the *target* of E. The notation xEy means $(x, y) \in E$. For $x \in X$, the set $E(x) = \{ y \in Y \mid xEy \}$ (which could be empty) is called the *target fiber* of E over x. The relation E can be identified to its target fiber map $x \in X \mapsto E(x) \in 2^Y$. More generally, the notation $E(S) = \bigcup_{x \in S} E(x)$ will be used for each $S \subset X$. The target fiber map can also be used to realize $E(S)$ as a subset of 2^Y; the context will clarify this ambiguity.

Definition 2.1. A relation, E, over two topological spaces, X and Y, is called *continuous* if the target fiber map $x \in X \mapsto E(x) \in 2^Y$ is continuous.

The following result, which incidentally is [12, Proposition 2.1], follows directly from (1).

Lemma 2.2. *A relation $E \subset X \times Y$ is continuous if and only if $\{x \in X \mid E(x) \subset F\}$ is closed in X for any closed $F \subset Y$.*

Let π_X and π_Y denote the factor projections of $X \times Y$ onto X and Y, respectively. If $A \subset X$, $B \subset Y$, and $x \in X$, then

$$A \cap E^{-1}(P_B) = \pi_X(E \cap (A \times B)) \;, \tag{2}$$

$$E(x) = \pi_Y(E \cap (\{x\} \times Y)) \;. \tag{3}$$

The following lemma is an easy consequence of (2).

Lemma 2.3. *A relation $E \subset X \times Y$ is continuous if and only if the restriction $\pi_X|_E : E \to X$ is an open mapping.*

For a relation, E, over X and Y, the *opposite* of E is the relation E^{op} over Y and X given by

$$E^{\mathrm{op}} = \{ (y, x) \in Y \times X \mid xEy \} \;.$$

The target fibers of E^{op} are $E^{\mathrm{op}}(y) = E^{-1}(P_{\{y\}})$, and are called *source fibers* of E. Note that for all $A \subset X$ and all $B \subset Y$,

$$(E^{\mathrm{op}})^{-1}(P_A) = E(A) \;, \tag{4}$$

$$(E \cap (A \times B))^{\mathrm{op}} = E^{\mathrm{op}} \cap (B \times A) \;. \tag{5}$$

Because of (4), $E^{\mathrm{op}} : Y \to 2^X$ is continuous if and only if, for any open set $O \subset X$, the set $E(O)$ is open in Y. In the case of equivalence relations, it is usually said that E is open when this property is satisfied; this term is now generalized to arbitrary relations.

Definition 2.4. A relation, E, over topological spaces is called *open* if E^{op} is continuous, and it is called *bi-continuous* if it is continuous and open.

Relation E could also be open in the sense that the map $E : X \to 2^Y$ is open; this possible ambiguity will be clarified by the context.

If E is a relation over a space X, then the source and target fibers are equal, which are simply called *fibers* of E, and so E is bi-continuous if and only if E is continuous.

Example 2.5. The following are basic examples of continuous and bi-continuous relations.

(i) If E is the graph of a map $f : X \to Y$, then E (respectively, E^{op}) is continuous just when f is continuous (respectively, open). In particular, the diagonal $\Delta_X \subset X \times X$ is a bi-continuous relation over X because it is the graph of the identity map of X.

(ii) If $E \subset X \times Y$ is an open subset, then E is a bi-continuous relation over X and Y.

(iii) If E is a continuous relation over X and Y, then $E \cap (A \times V)$ is a continuous relation over A and V, for any $A \subset X$ and any open $V \subset Y$. Thus, by (5), if E is bi-continuous, then $E \cap (U \times V)$ is a bi-continuous relation over U and V, for all open subsets $U \subset X$ and $V \subset Y$.

(iv) An equivalence relation is bi-continuous just when the saturation of any open set is an open set. In particular, the equivalence relation defined by the orbits of a continuous group action is bi-continuous, and the equivalence relation defined by the leaves of any foliated space is also bi-continuous.

For any family of relations $E_i \subset X \times Y$, $i \in I$, and any $A \subset Y$, the following properties hold:

$$\Big(\bigcup_i E_i\Big)^{-1}(P_A) = \bigcup_i E_i^{-1}(P_A) \;, \tag{6}$$

$$\Big(\bigcap_i E_i\Big)^{-1}(P_A) \subset \bigcap_i E_i^{-1}(P_A) \;,$$

$$\Big(\bigcup_i E_i\Big)^{\mathrm{op}} = \bigcup_i E_i^{\mathrm{op}} \;, \tag{7}$$

$$\Big(\bigcap_i E_i\Big)^{\mathrm{op}} = \bigcap_i E_i^{\mathrm{op}} \;. \tag{8}$$

The following result is a direct consequence of (6) and (7).

Lemma 2.6. *If E_i, $i \in I$, is a continuous (respectively, bi-continuous) relation over X and Y, then $\bigcup_{i\in I} E_i$ is a continuous (respectively, bi-continuous) relation over X and Y.*

Remark 1. The intersection of two continuous relations is a relation that need not be continuous. For example, if R_1 and R_2 are the relations over $\mathbf{R}$ given by the graphs of two linear mappings $\mathbf{R} \to \mathbf{R}$ that have different slope, then the intersection $R = \mathbf{R}_1 \cap R_2$ is not a continuous relation. However, the intersection of two continuous relations is continuous when one of the relations is also an open subset (Example 2.5-(ii)), as the next lemma shows.

Lemma 2.7. *Let E be a continuous* (*respectively, bi-continuous*) *relation over X and Y, and let $F \subset X \times Y$ be an open subset. Then $E \cap F$ is continuous* (*respectively, bi-continuous*) *relation over X and Y.*

Proof. Suppose that E is continuous. Let $V \subset Y$ be an open set, and let $x \in (E \cap F)^{-1}(P_V)$. Then there is some $y \in (E \cap F)(x) \cap V = E(x) \cap F(x) \cap V$; since F is an open subset of $X \times Y$ that contains (x, y), there are open sets $U \subset X$ and $W \subset Y$ such that $(x, y) \in U \times W \subset F$. By Example 2.5-(iii), $E \cap (U \times W)$ is a continuous relation over U and W, and so $(E \cap (U \times W))^{-1}(P_V)$ is open in U, hence in X. Since $x \in (E \cap (U \times W))^{-1}(P_V) \subset (E \cap F)^{-1}(P_V)$, this shows that $(E \cap F)^{-1}(P_V)$ is open in X, and hence that $E \cap F$ is a continuous relation.

If E is a bi-continuous relation, then $E \cap F$ is a bi-continuous relation because of Example 2.5-(ii) and (8). ☐

The *composition* of two relations, $E \subset X \times Y$ and $F \subset Y \times Z$, is the relation $F \circ E \subset X \times Z$ given by

$$F \circ E = \{ (x, z) \in X \times Z \mid \exists y \in Y \text{ such that } xEy \text{ and } yFz \} .$$

Composition of relations is an associative operation and Δ_X is its identity at X. Moreover

$$(F \circ E)^{\mathrm{op}} = E^{\mathrm{op}} \circ F^{\mathrm{op}} . \tag{9}$$

If $E \subset X \times X$ is a relation, the symbol E^n, for positive $n \in \mathbf{N}$, denotes the n-fold composition $E \circ \cdots \circ E$, and $E^0 = \Delta_X$ denotes the identity relation. If $E' \subset X' \times Y'$ is another relation over topological spaces, let $E \times E'$ be the relation over $X \times X'$ and $Y \times Y'$ given by

$$E \times E' = \{ (x, x', y, y') \in X \times X' \times Y \times Y' \mid xEy \text{ and } x'E'y' \} .$$

Note that

$$(E \times E')^{\mathrm{op}} = E^{\mathrm{op}} \times E'^{\mathrm{op}} . \tag{10}$$

For relations $E \subset X \times Y$ and $G \subset X \times Z$, let (E, G) denote the relation over X and $Y \times Z$ given by

$$(E, G) = \{ (x, y, z) \in X \times Y \times Z \mid xEy \text{ and } xGz \} .$$

Lemma 2.8. *The following properties hold:*

(*i*) *If E and F are continuous* (*respectively, bi-continuous*) *relations, then $F \circ E$ is also continuous* (*respectively, bi-continuous*) *relation.*

(*ii*) *If E and E' are continuous* (*respectively, bi-continuous*) *relations, then $E \times E'$ is a continuous* (*respectively, bi-continuous*) *relation.*

(*iii*) *If E and G are continuous relations, then (E, G) is a continuous relation.*

Proof. In (i) and (ii), the statements about continuity hold because

$$(F \circ E)^{-1}(P_W) = E^{-1}\left(P_{F^{-1}(P_W)}\right),$$
$$(E \times E')^{-1}(P_{V \times V'}) = E^{-1}(P_V) \times E'^{-1}(P_{V'}),$$

for $W \subset Z$, $V \subset Y$ and $V' \subset Y'$; and the statements about bi-continuity follow from (9) and (10). Property (iii) is a consequence of (i) and (ii) since

$$(F, G) = (F \times G) \circ (\Delta_X, \Delta_X)$$

and (Δ_X, Δ_X) is continuous because it is the graph of the diagonal mapping $x \mapsto (x, x)$. $\square$

Because of Lemma 2.8-(i), the continuous relations (and also the bi-continuous relations) over topological spaces are the morphisms of a category with the operation of composition. The assignment $E \mapsto E^{\mathrm{op}}$ is a contravariant functor of the category of bi-continuous relations to itself.

Lemma 2.9. *Let X be a topological space and let Y be a second countable topological space. The following properties are true.*

(*i*) *If $E \subset X \times Y$ is a continuous relation, then*

$$\{\, x \in X \mid E(x) \text{ is dense in } Y \,\}$$

is a G_δ subset of X.

(*ii*) *If $E, F \subset X \times Y$ are continuous relations and $E \subset F$, then*

$$\{\, x \in X \mid E(x) \text{ is dense in } F(x) \,\}$$

is a Borel subset of X.

Proof. Let $\mathcal{B}$ be any countable base of non-empty open sets for the topology of Y. Property (i) is true because

$$\{\, x \in X \mid E(x) \text{ is dense in } Y \,\} = \bigcap_{U \in \mathcal{B}} E^{-1}(P_U),$$

and Property (ii) is true because

$$\begin{aligned}\{\, x \in X \mid E(x) \text{ is dense in } F(x) \,\} &= \bigcap_{U \in \mathcal{B}} \{\, x \in X \mid x \in F^{-1}(P_U) \Rightarrow x \in E^{-1}(P_U) \,\} \\ &= \bigcap_{U \in \mathcal{B}} \left(E^{-1}(P_U) \cup (X \setminus F^{-1}(P_U))\right). \quad \square\end{aligned}$$

Definition 2.10. An equivalence relation over a topological space is called (*topologically transitive* (respectively, *topologically minimal*) if some equivalence class is dense (respectively, every equivalence class is dense).

The following concepts and notation will be used frequently.

Definition 2.11. (i) A subset of a topological space is meager if it is the countable intersection of nowhere dense subsets.

(ii) A subset of a topological space is residual if it contains the intersection of a countable family of dense open subsets.

(iii) A topological space is Baire if every residual subset is dense.

Definition 2.12. Let P be a property that members of sets may or may not have. Let X be a topological space.

(i) Property P is satisfied for residually many members of a topological space, X, and denoted by $(\forall^* x \in X)P(x)$, if the set $\{ x \in X \mid P(x) \}$ is residual in X.

(ii) Property P is satisfied for all but meagerly many members of X, and denoted by $(\exists^* x \in X)P(x)$, if the set $\{ x \in X \mid P(x)\}$ is non-meager.

Corollary 2.13. *If X is second countable and E is a topologically transitive, continuous equivalence relation over X, then $E(x)$ is dense in X, $\forall^* x \in X$.*

Proof. By Lemma 2.9-(i), the set

$$\{ x \in X \mid E(x) \text{ is dense in } X \}$$

is a dense G_δ subset of X. ☐

Lemma 2.14. *Let X be a topological space, let Y be a second countable topological space, and let $E \subset X \times Y$ be a continuous relation. If every source fiber of E is a Baire space, then the following properties hold:*

(*i*) *If A is a G_δ subset of Y, then*

$$\{ x \in X \mid E(x) \cap A \textit{ is residual in } E(x) \}$$

is a G_δ subset of X.

(*ii*) *If B is an F_σ subset of Y, then*

$$\{x \in X \mid E(x) \cap B \textit{ is non-meager in } E(x)\}$$

is an F_σ subset of X.

(*iii*) *If B is a Borel subset of Y, then*

$$\{ x \in X \mid E(x) \cap B \textit{ is residual in } E(x) \}$$

and

$$\{ x \in X \mid E(x) \cap B \textit{ is non-meager in } E(x) \}$$

are Borel subsets of X.

Proof. To prove (i), write $A = \bigcap_{n\in N} U_n$, where $\{U_n\}_{n\in N}$ is a countable family of open subsets of Y. For each $n \in N$, let $\mathcal{B}_n$ be a countable family of non-empty open subsets of U_n that is a base for the topology of U_n. Then

$$\begin{aligned}
&\{\, x \in X \mid E(x) \cap A \text{ is residual in } E(x) \,\} \\
&\qquad = \bigcap_{n\in N} \{\, x \in X \mid E(x) \cap U_n \text{ is residual in } E(x) \,\} \\
&\qquad = \bigcap_{n\in N} \{\, x \in X \mid E(x) \cap U_n \text{ is dense in } E(x) \,\} \\
&\qquad = \bigcap_{n\in N} \bigcap_{V\in\mathcal{B}_n} E^{-1}(P_V)
\end{aligned}$$

is a G_δ subset of X.

Property (ii) is a consequence of (i) because, by [6, Proposition 8.26],

$$\begin{aligned}
&\{\, x \in X \mid E(x) \cap B \text{ is non-meager in } E(x) \,\} \\
&\qquad = X \setminus \{\, x \in X \mid E(x) \cap (X \setminus B) \text{ is residual in } E(x) \,\} \,, \qquad (11)
\end{aligned}$$

for any $B \subset X$.

To prove (iii), let $\mathcal{C}$ be the collection of all subsets $B \subset Y$ such that, for any open subset $U \subset Y$, the sets

$$\{\, x \in X \mid E(x) \cap U \cap B \text{ is residual in } E(x) \cap U \,\}$$

and

$$\{\, x \in X \mid E(x) \cap U \cap B \text{ is non-meager in } E(x) \cap U \,\}$$

are both Borel subsets of X.

This collection $\mathcal{C}$ is a σ-algebra of subsets of X. Indeed, it is closed under complementation, because of (11) and Example 2.5-(iii), and it is also closed under countable intersections, because if $\{C_n \mid n \in \mathbf{N}\}$ is a countable family of members of $\mathcal{C}$, and $U \subset Y$ is any open set, then

$$\begin{aligned}
&\Big\{\, x \in X \mid E(x) \cap U \cap \bigcap_n C_n \text{ is residual in } E(x) \cap U \,\Big\} \\
&\qquad = \bigcap_n \{\, x \in X \mid E(x) \cap U \cap C_n \text{ is residual in } E(x) \cap U \,\}
\end{aligned}$$

is a Borel subset of X. Therefore, for any countable family $\mathcal{B}$ of open, non-empty, subsets of U that is a base for the topology of U, by [6, Proposition 8.26],

$$\begin{aligned}
&\Big\{\, x \in X \mid E(x) \cap U \cap \bigcap_n C_n \text{ is non-meager in } E(x) \cap U \,\Big\} \\
&\qquad = \bigcup_{V\in\mathcal{B}} \Big\{\, x \in X \mid E(x) \cap V \cap \bigcap_n C_n \text{ is residual in } E(x) \cap V \,\Big\} \\
&\qquad = \bigcup_{V\in\mathcal{B}} \bigcap_n \{\, x \in X \mid E(x) \cap V \cap C_n \text{ is residual in } E(x) \cap V \,\}
\end{aligned}$$

is a Borel subset of X, and so $\bigcap_n C_n \in \mathcal{C}$.

The collection $\mathcal{C}$ contains all the open subsets of X. Indeed, if $V \subset Y$ is any open set, then use Example 2.5-(iii), and apply (i) and [6, Proposition 8.26] to obtain that

$$\{x \in X \mid E(x) \cap U \cap V \text{ is non-meager in } E(x) \cap U\} = E^{-1}(P_{U\cap V}) \ .$$

Therefore $\mathcal{C}$ is a σ-algebra that contains all the open subsets of X, and thus it also contains all the Borel subsets of X, which establishes (iii). $\square$

Lemma 2.15. *Let $E \subset X \times Y$ be an open relation over X and Y. If $A \subset B \subset Y$ and A is dense in B, then $E^{-1}(P_A)$ is dense in $E^{-1}(P_B)$.*

Proof. Let O be an open subset of X. Because $E(O)$ is open in Y and A dense in B,

$$\begin{aligned} O \cap E^{-1}(P_B) \neq \emptyset &\Longleftrightarrow E(O) \cap B \neq \emptyset \\ &\Longrightarrow E(O) \cap A \neq \emptyset \Longleftrightarrow O \cap E^{-1}(P_A) \neq \emptyset \ . \quad \square \end{aligned}$$

Lemma 2.16. *Let E be a bi-continuous relation over the topological spaces X and Y, and assume that Y is second countable. If B is open and dense in Y, then $B \cap E(x)$ is open and dense in $E(x)$ $\forall^* x \in X$.*

Proof. Let $\{U_n\}_{n\in N}$ be a countable base for the topology of Y. Write

$$O_n = \big(X \setminus E^{-1}(P_{U_n})\big) \cup E^{-1}(P_{U_n\cap B}) \ .$$

The boundary $\partial E^{-1}(P_{U_n})$ is a meager set in X because $E^{-1}(P_{U_n})$ is open in X. Since $U_n \cap B$ is dense in U_n, Lemma 2.15 implies that $E^{-1}(P_{U_n\cap B})$ is dense in $E^{-1}(P_{U_n})$. Hence

$$\big(X \setminus \overline{E^{-1}(P_{U_n})}\big) \cup E^{-1}(P_{U_n\cap B})$$

is open and dense in $X \setminus \partial E^{-1}(P_{U_n})$, and therefore the interior of O_n is open and dense in X. This proves that $\bigcap_n O_n$ is a residual subset of X. If x is in $\bigcap_{n\in N} O_n$, then $E(x) \cap B$ is dense in $E(x)$, for otherwise there would be some n in $\mathbf{N}$ such that $E(x)\cap B\cap U_n = \emptyset$ and $E(x)\cap U_n \neq \emptyset$, which conflicts with the definition of O_n. $\square$

The following result is a generalization of the Kuratowski-Ulam Theorem [**?**, p. 222].

Theorem 2.17. *Let X and Y be topological spaces, with Y second countable, and let E be a bi-continuous relation over X and Y. The following are true:*

(*i*) *if $A \subset Y$ has the Baire property, then $A \cap E(x)$ has the Baire property in $E(x)$ $\forall^* x \in X$;*

(*ii*) *if A is meager in Y, then $A \cap E(x)$ is meager in $E(x)$ $\forall^* x \in X$;*

(*iii*) *if A is residual in Y, then $A \cap E(x)$ is residual in $E(x)$ $\forall^* x \in X$.*

Furthermore, if $E(x)$ is dense in Y and if $E(x)$ is a Baire space for residually many $x \in X$, then the converses to (*ii*) *and* (*iii*) *are also true.*

Proof. Lemma 2.16 implies (iii), which in turn implies (ii).

To prove (i), suppose that $A \subset Y$ has the Baire property. This means that $A = U \triangle M$ for some meager set $M \subset Y$ and some open set $U \subset Y$. So

$$A \cap E(x) = \big(U \cap E(x)\big) \triangle \big(M \cap E(x)\big)$$

for all $x \in X$. Here, $U \cap E(x)$ is open in $E(x)$, and $M \cap E(x)$ is meager in $E(x)$ $\forall^* x \in X$ by (ii).

Assume next that $E(X)$ is dense in Y and that $E(x)$ is a Baire space $\forall^* x \in X$. Let A be a non-meager subset of Y with the Baire property. Because of [6, Proposition 8.26], there is a non-empty open $U \subset Y$ such that $A \cap U$ is residual in U; hence, by (iii), $A \cap U \cap E(x)$ is residual in $U \cap E(x)$ $\forall^* x \in X$. Because of [6, 8.22], $A \cap U$ has the Baire property in X, and thus in U; hence, by (i), $A \cap U \cap E(x)$ has the Baire property in $U \cap E(x)$ $\forall^* x \in X$. Because E is continuous and $E(X)$ is dense in Y, $E^{-1}(P_U)$ is an open non-empty subset of X. Since $E(x)$ is also a Baire space $\forall^* x \in X$, it follows from [6, Proposition 8.26] that $A \cap E(x)$ is not meager in $E(x)$ $\forall^* x \in E^{-1}(P_U)$. Thus $\exists^* x \in X$ such that $A \cap E(x)$ is not meager in $E(x)$. This proves the converse of (ii), which in turn implies the converse of (iii). □

Remark 2. The classical Kuratovski-Ulam Theorem (*loc. cit.*, *cf.* also [6, Theorem 8.41]) is obtained from Theorem 2.17 by taking $X = Y = X_1 \times X_2$, where X_1 and X_2 are second countable spaces, and E equal to the equivalence relation whose equivalence classes are the fibers $\{x_1\} \times X_2$ for $x_1 \in X_1$.

Corollary 2.18. *Let X and Y be second countable topological spaces, and let $A, E \subset X \times Y$. Suppose that E is a bi-continuous relation whose source and target fibers are Baire spaces. Then $(x, y) \in A$ $\forall^* y \in E(x)$ $\forall^* x \in X$ if and only if $(x, y) \in A$ $\forall^* x \in E^{\mathrm{op}}(y)$ $\forall^* y \in Y$.*

Proof. Lemma 2.3 implies that the restrictions of the projections π_X and π_Y to E are open mappings. Hence, by Example 2.5-(i), their corresponding graphs, $\Pi_{E,X} \subset E \times X$ and $\Pi_{E,Y} \subset E \times Y$, are bi-continuous relations. Moreover, for $x \in X$ and $y \in Y$,

$$\begin{aligned} \Pi^{\mathrm{op}}_{E,X}(x) &= \{x\} \times E(x)\ , & \Pi^{\mathrm{op}}_{E,Y}(y) &= E^{\mathrm{op}}(y) \times \{y\}\ , \\ A \cap \Pi^{\mathrm{op}}_{E,X}(x) &= \{x\} \times (A \cap E)(x)\ , & A \cap \Pi^{\mathrm{op}}_{E,Y}(y) &= (A \cap E)^{\mathrm{op}}(y) \times \{y\}\ . \end{aligned}$$

Then, by Theorem 2.17,

$$\begin{aligned} &(x, y) \in A \forall^* y \in E(x)\ \forall^* x \in X \\ &\qquad \iff (A \cap E)(x) \text{ is residual in } E(x)\ \forall^* x \in X \\ &\qquad \iff A \cap E \text{ is residual in } E \\ &\qquad \iff (A \cap E)^{\mathrm{op}}(y) \text{ is residual in } E^{\mathrm{op}}(y)\ \forall^* y \in Y \\ &\qquad \iff (x, y) \in A\ \forall^* x \in E^{\mathrm{op}}(y)\ \forall^* y \in Y\ . \quad \square \end{aligned}$$

Corollary 2.19. *The following properties hold:*

(*i*) *Let X and Y be second countable topological spaces, and let $E_n \subset X \times Y$ be a bi-continuous relation for each $n \in \mathbf{N}$. If $A \subset X$ and $B \subset Y$ are residual subsets, then there are residual subsets $C \subset A$ and $D \subset B$ such that $D \cap E_n(x)$ is residual in $E_n(x)$ for all $x \in C$ and all $n \in \mathbf{N}$, and $C \cap E_n^{\mathrm{op}}(y)$ is residual in $E_n^{\mathrm{op}}(y)$ for all $y \in D$ and all $n \in \mathbf{N}$.*

(*ii*) *Let X be a second countable topological space, and $E_n \subset X \times X$ a bi-continuous relation for each $n \in \mathbf{N}$. If $A \subset X$ is a residual subset, then there is some residual subset $C \subset A$ such that $C \cap E_n(x)$ is residual in $E_n(x)$ for all $x \in C$ and all $n \in \mathbf{N}$.*

Proof. To prove (i), define sequences of residual subsets, $C_i \subset X$ and $D_i \subset Y$, by the following induction process on $i \in \mathbf{N}$. Set $C_0 = A$ and $D_0 = B$. Assuming that C_i and D_i have been defined, let

$$C_{i+1} = \{\, x \in X \mid D_i \cap E_n(x) \text{ is residual in } E_n(x)\ \forall^* x \in X\ \&\ \forall n \in \mathbf{N} \,\}\ ,$$

and

$$D_{i+1} = \{\, y \in Y \mid C_i \cap E_n^{\mathrm{op}}(y) \text{ is residual in } E_n^{\mathrm{op}}(y)\ \forall^* y \in Y\ \&\ \forall n \in \mathbf{N} \,\}\ .$$

By Theorem 2.17, C_i is residual in X and D_i is residual in Y, for all $i \in \mathbf{N}$, and therefore $C = \bigcap_{i \in \mathbf{N}} C_i$ is residual in A and $D = \bigcap_{i \in \mathbf{N}} D_i$ is residual in B. Moreover, for all $n \in \mathbf{N}$, $D \cap E_n(x) = \bigcap_{i \in \mathbf{N}} (D_i \cap E_n(x))$ is residual in $E_n(x)$ for all $x \in C$, and $C \cap E_n^{\mathrm{op}}(y) = \bigcap_i (C_i \cap E_n^{\mathrm{op}}(y))$, is residual in $E_n^{\mathrm{op}}(y)$, for all $y \in D$.

To prove (ii), let $C_0 = A$ and, assuming that C_i has been defined, let

$$C_{i+1} = \{\, x \in X \mid C_i \cap E_n(x) \text{ is residual in } E(x)\ \forall^* x \in X\ \&\ \forall n \in \mathbf{N} \,\}\ .$$

By Theorem 2.17, C_i is residual in X, for all $i \in \mathbf{N}$. Therefore $C = \bigcap_{i \in \mathbf{N}} C_i$ is residual in A, and $C \cap E_n(x) = \bigcap_{i \in \mathbf{N}} (C_i \cap E_n(x))$ is residual in $E_n(x)$, for all $x \in C$ and all $n \in \mathbf{N}$. $\square$

3 Classification and generic ergodicity

Let X and Y be topological spaces, and let $E \subset X \times X$ and $F \subset Y \times Y$ be equivalence relations. A mapping, $\theta : X \to Y$, is called (E, F)-*invariant* if

$$xEx' \Longrightarrow \theta(x) F \theta(x')$$

for all $x, x' \in X$. Such (E, F)-invariant mapping θ induces a mapping, denoted by $\bar{\theta} : X/E \to Y/F$, between the corresponding quotient spaces.

The relation E is said to be *Borel reducible* to F, denoted by $E \leq_B F$, if there is an (E, F)-invariant Borel mapping $\theta : X \to Y$ such that the induced mapping $\bar{\theta} : X/E \to Y/F$ is injective. If $E \leq_B F$ and $F \leq_B E$, then E is said to be *Borel bi-reducible* with F, and is denoted by $E \sim_B F$.

The relation E is said to be *generically F-ergodic* if, for any (E, F)-invariant, Baire measurable mapping $\theta : X \to Y$, there is some residual saturated $C \subset X$ such that $\bar{\theta} : C/(E \cap (C \times C)) \to Y/F$ is constant.

Remark 3. If E is a generically F-ergodic relation over X, then any equivalence relation over X that contains E is also generically F-ergodic.

The partial pre-order relation $\leq_B$ establishes a hierarchy on the complexity of equivalence relations over topological spaces. Two key ranks of this hierarchy are given by the following two concepts of classification of relations. In the first one, E is said to be *concretely classifiable* (or *smooth*, or *tame*) if $E \leq_B \Delta_{\mathbf{R}}$ (recall that $\Delta_{\mathbf{R}} \subset \mathbf{R} \times \mathbf{R}$ denotes the diagonal). This means that the equivalence classes of E can be distinguished by some Borel mapping $X \to \mathbf{R}$.

Theorem 3.1. *Let X and Y be second countable topological spaces. If E is a continuous, topologically transitive equivalence relation over X, then E is generically Δ_Y-ergodic.*

Proof. Let $\theta : X \to Y$ be (E, Δ_Y)-invariant and Baire measurable. By [6, Theorem 8.38], θ is continuous on some residual saturated set $C_0 \subset X$. By Corollary 2.13, there is residual saturated $C_1 \subset X$ such that $E(x)$ is dense in X, for all $x \in C_1$. Then $C_0 \cap C_1$ is a residual subset of X where θ is constant. □

Remark 4. In the above proof, if X is a Baire space, then $C_0 \cap C_1 \neq \emptyset$.

Corollary 3.2. *Let X be a second countable space and let E be a continuous equivalence relation over X. If E is topologically transitive, then any E-saturated subset of X that has the Baire property is either residual or meager.*

Proof. For any saturated subset of X with the Baire property, apply Theorem 3.1 to its characteristic function $X \to \{0, 1\}$. □

Corollary 3.3. *Let X be a second countable Baire space and let E be a continuous equivalence relation over X. If E is topologically transitive and its equivalence classes are meager subsets of X, then E is not concretely classifiable.*

Proof. By Theorem 3.1, each $(E, \Delta_{\mathbf{R}})$-invariant Borel map $\theta : X \to \mathbf{R}$ is constant on some residual saturated subset of X. So $\bar{\theta} : X/E \to \mathbf{R}/\Delta_{\mathbf{R}} \equiv \mathbf{R}$ cannot be injective because X is a Baire space and the equivalence classes are meager. □

The second classification concept can be defined by using $\prod_{n=1}^{\infty} \mathbf{2}^{\mathbf{N}^n}$ endowed with the product topology, which is a Polish space. Each element of $\prod_{n=1}^{\infty} \mathbf{2}^{\mathbf{N}^n}$ can be considered as a structure on $\mathbf{N}$ defined by a sequence (R_n), where each R_n is a relation over $\mathbf{N}$ with arity n. Two such structures are isomorphic when they correspond by some permutation of $\mathbf{N}$, which defines the isomorphism relation $\cong$ over $\prod_{n=1}^{\infty} \mathbf{2}^{\mathbf{N}^n}$. Then a relation E is *classifiable by countable structures* (or *models*) if $E \leq_B \cong$. This means that there is some Borel map $\theta : X \to \prod_{n=1}^{\infty} 2^{\mathbf{N}^n}$ such that xEx' if and only if $\theta(x) \cong \theta(x')$. Here, it is also possible to use the structures on $\mathbf{N}$ defined by arbitrary countable relational languages, *cf.* [5, Section 2.3].

The equivalence relation defined by the action of a group G on a set X will be denoted by E_G^X; in this case, the notation $\mathcal{O}(x)$ will be used for the orbit of each $x \in X$ instead of $E_G^X(x)$. If G is a Polish group, the family of all relations defined by continuous actions of G on Polish spaces has a maximum with respect to $\leq_B$, which is unique up to $\sim_B$ and is denoted by E_G^∞ [1, 8].

As a special example, the group S_∞ of permutations of $\mathbf{N}$ becomes Polish with the topology induced by the product topology of $\mathbf{N}^{\mathbf{N}}$, where $\mathbf{N}$ is considered with the discrete topology. Then the canonical action of S_∞ on $\prod_{n=1}^{\infty} \mathbf{2}^{\mathbf{N}^n}$ defines the isomorphism relation $\cong$ over the space of countable structures, which is a representative of $E^{\infty}_{S_\infty}$ [5].

Classification by countable structures and generic ergodicity are well understood for equivalence relations defined by Polish actions in terms of a dynamical concept called *turbulence* which was introduced by Hjorth [5].

4 Turbulent uniform relations

A *uniform equivalence relation*, or simply a *uniform relation*, over a set, X, is a pair, $(\mathcal{V}, E)$, consisting of a uniformity $\mathcal{V}$ on X and an equivalence relation E over X such that $E \in \mathcal{V}$. Note that $(\mathcal{V}, E)$ is determined by the entourages (members of $\mathcal{V}$) that are contained in E, and that $\mathcal{V}$ induces a uniform structure on each equivalence class of E.

One important example of a uniform relation is that given by the action of a topological group, G, on a set, X. This is of the form $(\mathcal{V}, E^X_G)$, where $\mathcal{V}$ is the uniform structure on X generated by the entourages

$$V_W = \{\, (x, gx) \mid x \in X \;\&\; g \in W \,\}\ , \tag{12}$$

where W belongs to the neighborhood system of the identity of G. Thus a uniform relation over a topological space can be considered as a generalized dynamical system.

Another important example of uniform relation is the following. A *metric* (or *distance function*) *with possible infinite values* on a set is a function $d : X \times X \to [0, \infty]$ satisfying the usual properties of a metric (d vanishes just on the diagonal of $X \times X$, is symmetric and satisfies the triangle inequality). It defines an equivalence relation over X denoted by E^X_d and given by $x\, E^X_d\, y$ if and only if $d(x, y) < \infty$. There is a uniform relation induced by d of the form $(\mathcal{V}, E^X_d)$, where a base of $\mathcal{V}$ consists of the entourages

$$V_\epsilon = \{\, (x, y) \in X \times X \mid d(x, y) < \epsilon \,\}\ . \tag{13}$$

The term *metric equivalence relation* (or *metric relation*) will be used for the pair (d, E^X_d) (or even for d). Like the usual metrics, metrics with possible infinite values induce a topology which has a base of open sets consisting of open balls; unless otherwise indicated, the ball of center x and radius R will be denoted by $B_X(x, R)$ or $B_d(x, R)$, or simply by $B(x, R)$.

Remark 5. Other generalizations of metrics also define uniform relations, like *pseudo-metrics with possible infinite values*, defined in the obvious way, or when the triangle inequality is replaced by the condition $d(x, y) \le \rho\big(d(x, z) + d(z, y)\big)$ for some $\rho > 0$ and all $x, y, z \in X$ (*generalized pseudo-metrics with possible infinite values*). They give rise to the concepts of *pseudo-metric relation* and *generalized pseudo-metric relation*.

Remark 6. Let d and d' be metric relations over X that induce respective uniform relations $(\mathcal{V}, E)$ and $(\mathcal{V}', E')$. If $d' \le d$, then $\mathcal{V} \subset \mathcal{V}'$ and $E \subset E'$.

Definition 4.1. Let $(\mathcal{V}, E)$ be a uniform relation over a topological space X. For any non-empty open $U \subset X$ and any $V \in \mathcal{V}$ with $V \subset E$, the set

$$E(U, V) = \bigcup_{n=0}^{\infty} (V \cap (U \times U))^n$$

is an equivalence relation over U called a *local equivalence relation*. The $E(U, V)$-equivalence class of any $x \in U$ is called a *local equivalence class* of x, and denoted by $E(x, U, V)$.

For a relation given by the action of a group G on a space X, the local equivalence classes are called *local orbits* in Hjorth [5], and the notation $\mathcal{O}(x, U, W)$ is used instead of $E_G^X(x, U, V)$ when $V = V_W$ according to (12). Similarly, for a uniform relation induced by a generalized pseudo-metric d on a set X, the notation $E_d^X(x, U, \epsilon)$ is used instead of $E_d^X(x, U, V)$ when $V = V_\epsilon$ according to (13).

Definition 4.2. A uniform relation is called *turbulent* if:

(i) every equivalence class is dense,

(ii) every equivalence class is meager, and

(iii) every local equivalence class is somewhere dense.

Remark 7. Definition 4.2 does not correspond exactly to the definition of turbulence introduced by Hjorth for Polish actions [5, Definition 3.12]. To generalize exactly Hjorth's definition, condition (iii) of Definition 4.2 should be replaced with condition (iii'):

(iii') every equivalence class meets the closure of each local equivalence class.

In fact, (i) already follows from (iii'). In the case of Polish actions, (iii) and (iii') can be interchanged in the definition of turbulence by [5, Lemmas 3.14 and 3.16]; thus Definition 4.2 generalizes Hjorth's definition. But in our setting, that equivalence is more delicate and our results become simpler by using (iii).

Remark 8. Let $(\mathcal{V}, E)$ and $(\mathcal{V}', E')$ be uniform relations over a topological space X such that $\mathcal{V} \subset \mathcal{V}'$ and $E \subset E'$. If the local equivalence equivalence classes of $(\mathcal{V}, E)$ are somewhere dense (Definition 4.2-(iii)), then the local equivalence equivalence classes of $(\mathcal{V}', E')$ are also somewhere dense.

Example 4.3. The following simple examples illustrate the generalization of the concept of turbulence for uniform relations.

(i) If E is an equivalence relation over a topological space X, then the family $\mathcal{V} = \{\, V \subset X \times X \mid E \subset V \,\}$ is a uniformity on X, and $(\mathcal{V}, E)$ is a uniform relation. Therefore E is the only entourage of $\mathcal{V}$ contained in E, and $E(x, U, E) = E(x) \cap U$ for any open $U \subset X$ and all $x \in U$, so it follows that $(\mathcal{V}, E)$ is turbulent if the equivalence classes of E are dense and meager.

(ii) Let G be a first countable topological group whose topology is induced by a right invariant metric d_G. Suppose that G acts continuously on the left on a topological space X. Then this action induces a pseudo-metric relation d on X with $E_d^X = E_G^X$ and

$$d(x,y) = \inf\{\, d_G(1_G, g) \mid g \in G \,\&\, gx = y \,\}$$

for $(x,y) \in E_G^X$, where 1_G denotes the identity element of G. The pseudo-metric relation d induces the same uniform relation as the action of G on X, and therefore d is turbulent if and only the action is turbulent.

(iii) Let $\mathbf{Z}$ be the additive group of integers with the discrete topology, and let $G \subset \mathbf{Z}^{\mathbf{N}}$ denote the topological subgroup consisting of the sequences (x_n) such that $x_n = 0$ for all but finitely many $n \in \mathbf{N}$. For some fixed irrational number θ, consider the continuous action of G on the circle $S^1 \equiv \mathbf{R}/\mathbf{Z}$ given by $(x_n)\cdot[r] = [r + \theta \sum_n x_n]$, where $[r]$ is the element of S^1 represented by $r \in \mathbf{R}$. The orbits of this action are dense and countable. For each $N \in \mathbf{N}$, the sets

$$W_N = \{\, (x_n) \in G \mid x_n = 0\ \forall n \in \{0, \dots, N\} \,\}$$

are clopen subgroups of G which form a base of neighborhoods of the identity element. The induced action of each W_N on S^1 has the same orbits as G; so $\mathcal{O}([r], U, W_N) = U \cap \mathcal{O}([r])$ for all open $U \subset S^1$ and each $[r] \in U$. It follows that this action is turbulent. In fact, the uniform equivalence relation induced by this action is of the type described in (i): we have $E_G^{S^1} \subset V$ for each entourage V. Moreover, for any invariant metric on G, the induced pseudo-metric relation d on S^1 is determined by $d([r],[s]) = \infty$ if $\mathcal{O}([r]) \neq \mathcal{O}([s])$ and $d([r],[s]) = 0$ if $\mathcal{O}([r]) = \mathcal{O}([s])$. However, the action of G on S^1 given by $(x_n)\cdot[r] = [r + \theta x_0]$ has the same orbits but is not turbulent: each point is a local orbit. Indeed this second action induces the same uniform equivalence relation as the action of $\mathbf{Z}$ given by $x \cdot [r] = [r + \theta x]$, which is not turbulent because $\mathbf{Z}$ is locally compact.

Definition 4.4. A uniform relation, $(\mathcal{V}, E)$, on a space, X, is *generically turbulent* if:

(i) the equivalence class of x is dense in X $\forall^* x \in X$,

(ii) every equivalence class is meager, and

(iii) any local equivalence class of x is somewhere dense $\forall^* x \in X$.

5 Turbulence and generic ergodicity

From now on, only metric relations over topological spaces will be considered because that suffices for the applications given in this paper. Some restriction on the topological structure of the space, and some compatibility of that structure with the metric relation will be required, and these are given in the following definition; they are restrictive enough to prove the desired results, and general enough to be satisfied in the applications.

Definition 5.1. A metric relation, d, over a topological space, X, is said to be of *type I* if:

(i) X is Polish;

(ii) the topology induced by d on X is finer or equal than the topology of X; and

(iii) there is a family $\mathcal{E}$ of relations over X such that:

  (a) each $E \in \mathcal{E}$ is symmetric,

  (b) each $E \in \mathcal{E}$ is a G_δ subset of $X \times X$,

  (c) for each $r > 0$, there are some $E, F \in \mathcal{E}$ so that

$$E(x) \subset B_d(x, r) \subset F(x)$$

  for all $x \in X$,

  (d) for each $E \in \mathcal{E}$, there are some $r, s > 0$ so that

$$B_d(x, r) \subset E(x) \subset B_d(x, s)$$

  for all $x \in X$,

  (e) each $E \in \mathcal{E}$ is continuous, and

  (f) for all $E, F, G \in \mathcal{E}$ and $x \in X$, if $E \circ F \supset G$, then $E \cap (F(x) \times G(x))$ is an open relation over $F(x)$ and $G(x)$.

Remark 9. In Definition 5.1, observe the following:

(i) The family $\mathcal{E}$ can be chosen to be countable and completely ordered by inclusion; that is, $\mathcal{E} = \{E_n \mid n \in \mathbf{Z}\}$ so that $E_m \subset E_n$ if $m \leq n$.

(ii) Each $E \in \mathcal{E}$ is a G_δ subset of X and, for each $x \in X$, $E(x) \equiv E \cap (\{x\} \times X)$ is a G_δ subset of $X \equiv X \times \{x\}$. Therefore, by [6, Theorem 3.11], E and and $E(x)$ are Polish subspaces of $X \times X$ and X, respectively; in particular, they are Baire spaces.

(iii) Since $E_d^X = \bigcup_{E \in \mathcal{E}} E$, a metric relation of type I is continuous, by Lemma 2.6; however, its fibers need not be Polish spaces.

(iv) By properties (iii)-(a),(f), for all $E, F, G \in \mathcal{E}$ and $x \in X$, if $E \circ G \supset F$, then $E \cap (F(x) \times G(x))$ is a continuous relation over $F(x)$ and $G(x)$.

(v) It will become clear that the general results presented in this paper hold if the metric equivalence relation is of type I only on some dense G_δ subset. For the sake of simplicity, that generality is avoided since the conditions of Definition 5.1 are satisfied in applications to be given.

Lemma 5.2. *Let d be a metric relation of type I over a space X, let $\mathcal{E} = \{ E_n \mid n \in \mathbf{Z} \}$ be a family of subsets of $X \times X$ satisfying the conditions of Definition 5.1 and Remark 9-(i). Let G be a Polish group and let Y be a Polish G-space. If $\theta : X \to Y$ is an (E_d^X, E_G^Y)-invariant Borel map, then, for any neighborhood W of the identity element 1_G in G, $\forall \ell \in \mathbf{Z}$, $\forall^* x \in X$, and $\forall^* x' \in E_\ell(x)$, there is some open neighborhood U of x in X such that, $\forall k \in \mathbf{Z}$ and $\forall x'' \in U \cap E_k(x) \cap E_\ell(x')$, $\exists g \in W$ so that $g \cdot \theta(x) = \theta(x'')$.*

Proof. Fix an open neighborhood W of 1_G in G. The result follows from Corollary 2.18 and the following Claim 1.

Claim 1. $\forall \ell \in \mathbf{Z}$, $\forall x \in X$ and $\forall^* x' \in E_\ell(x)$, there exists some open neighborhood U of x' in X such that, $\forall k \in \mathbf{Z}$ and $\forall^* x'' \in U \cap E_k(x') \cap E_\ell(x)$, $\exists g \in W$ so that $g \cdot \theta(x') = \theta(x'')$.

To prove this claim, let W' be a symmetric open neighborhood of the identity $1_G \in G$ such that ${W'}^2 \subset W$. Since G is a Polish group, there are countably many elements $g_i \in G$, $i \in \mathbf{N}$, such that $G \subset \bigcup_{i \in \mathbf{N}} W' g_i$. Therefore, given $\ell \in \mathbf{Z}$ and $x \in X$, the set $\theta(E_\ell(x)) \subset \bigcup_{i \in \mathbf{N}} W' g_i \cdot \theta(x)$. The preimage of $W' g_i \cdot \theta(x)$ via the mapping $\theta : E_\ell(x) \to Y$ is analytic in $E_\ell(x)$ because $W' g_i \cdot \theta(x)$ is analytic [6, Proposition 14.4-(ii)]. Hence it has the Baire property [6, Theorem 21.6], and so there are open subsets $O_i \subset E_\ell(x)$ and residual subsets $C_i \subset O_i$ such that $\bigcup_i O_i$ is dense in $E_\ell(x)$ and $\theta(C_i) \subset W' g_i \cdot \theta(x)$. By using Definition 5.1-(iii)-(f) and Remark 9-(iv) applied to the relation $E_k \cap (E_\ell(x) \times E_\ell(x))$ over $E_\ell(x)$, and by Corollary 2.19-(ii) and Example 2.5-(iii), it follows that there is some residual $D_i \subset C_i$ such that $E_k(x') \cap D_i$ is residual in $E_k(x') \cap O_i$ for all $x' \in D_i$ and $k \in \mathbf{Z}$.

The union $A = \bigcup_i D_i$ is residual in $E_\ell(x)$. If $x' \in A$, then $x \in D_i$ for some i and so $\theta(x') = g' g_i \cdot \theta(x)$ for some $g' \in W'$. Let U be any open neighborhood of x' in X so that $U \cap E_\ell(x) \subset O_i$. Then $U \cap E_k(x') \cap D_i$ is residual in $U \cap E_k(x') \cap E_\ell(x)$ $\forall k \in \mathbf{N}$. Moreover, for each $x'' \in E_k(x') \cap D_i$, there is some $g'' \in W'$ so that $\theta(x'') = g'' g_i \cdot \theta(x)$. Therefore, if

$$g = g'' {g'}^{-1} \in W' {W'}^{-1} \subset W \ ,$$

then

$$g \cdot \theta(x') = g g' g_i \cdot \theta(x) = g'' g_i \cdot \theta(x) = \theta(x'') \ ,$$

which completes the proof of Claim 1. □

Corollary 5.3. *Under the conditions of Lemma 5.2, for any neighborhood W of the identity element 1_G of G and $\forall^* x \in X$, $\exists k \in \mathbf{Z}$ such that, $\forall^* x' \in E_k(x)$, $\exists g \in W$ so that $g \cdot \theta(x) = \theta(x')$.*

Proof. Fix any $\ell \in \mathbf{Z}$ and any open neighborhood W of 1_G in G. Then, $\forall^* x \in X$ and $\forall^* x' \in E_\ell(x)$, let U be an open neighborhood of x in X satisfying the statement of Lemma 5.2. By Definition 5.1-(ii),(iii)-(c) and Remark 9-(i), there is some $k \leq \ell$ so that $E_k(x) \subset U$, obtaining that, $\forall x'' \in E_k(x) \cap E_\ell(x')$, $\exists g \in W$ so that $g \cdot \theta(x) = \theta(x'')$. Then the result follows from Theorem 2.17, Definition 5.1-(iii)-(f) and Remark 9-(iv) with the relation $E_\ell \cap (E_\ell(x') \times E_k(x'))$ over $E_\ell(x')$ and $E_k(x')$. □

Theorem 5.4. *Let d be a metric relation of type I on a space X and let Y be a Polish S_∞-space. If there are residually many $x \in X$ for which any local equivalence class of x is somewhere dense, then E_d^X is generically $E_{S_\infty}^Y$-ergodic.*

Proof. Let $\theta : X \to Y$ be an $(E_d^X, E_{S_\infty}^Y)$-invariant Borel map. Consider a family of subsets of $X \times X$, $\mathcal{E} = \{ E_n \mid n \in \mathbf{Z} \}$, satisfying the conditions of Definition 5.1 and Remark 9-(i). The sets

$$W_N = \{ h \in S_\infty \mid h(\ell) = \ell \, \forall \ell \leq N \} ,$$

with $N \in \mathbf{N}$, form a base of neighbourhoods of the identity 1_{S_∞} in S_∞, which are clopen subgroups. Define $I : X \times \mathbf{N} \to \mathbf{N} \cup \{\infty\}$ by setting $I(x, N)$ equal to the least $\ell \in \mathbf{N}$ such that, $\forall^* x' \in E_{-\ell}(x)$, $\exists h \in W_N$ so that $h \cdot \theta(x) = \theta(x')$ if there is such an ℓ, and setting $I(x, N) = \infty$ if there is not such an ℓ. Let $\mathbf{N}$ and $\mathbf{N} \cup \{\infty\}$ be endowed with the discrete topologies.

Claim 2. I is Baire measurable.

The proof of Claim 2 is as follows. Let $\ell, N \in \mathbf{N}$. The set

$$S_N = \{ (y, h \cdot y) \mid y \in Y, \ h \in W_N \}$$

is analytic in $Y \times Y$, and $E_{-\ell}$ is a Polish space by Remark 9-(ii). So $R_{\ell,N} = E_{-\ell} \cap (\theta \times \theta)^{-1}(S_N)$ is analytic in $E_{-\ell}$ [6, Proposition 14.4-(ii)], and therefore $R_{\ell,N}$ has the Baire property [6, Theorem 21.6]. Hence there is some open $U_{\ell,N} \subset E_{-\ell}$ so that $R_{\ell,N} \triangle U_{\ell,N}$ is meager in $E_{-\ell}$. The restriction $E_{-\ell} \to X$ of the first factor projection $X \times X \to X$ is continuous and open by Lemma 2.3, so its graph $\Pi_\ell \subset E_{-\ell} \times X$ is a bi-continuous relation according to Example 2.5-(i). By Theorem 2.17-(ii), there is some residual $D_{\ell,N} \subset X$ such that $(R_{\ell,N} \triangle U_{\ell,N}) \cap \Pi_\ell^{\mathrm{op}}(x)$ is meager in $\Pi_\ell^{\mathrm{op}}(x)$ $\forall x \in D_{\ell,N}$. Notice that $\Pi_\ell^{\mathrm{op}}(x) = \{x\} \times E_{-\ell}(x) \equiv E_{-\ell}(x)$ and

$$\begin{aligned}(R_{\ell,N} \triangle U_{\ell,N}) \cap \Pi_\ell^{\mathrm{op}}(x) &= \{x\} \times (R_{\ell,N}(x) \triangle U_{\ell,N}(x)) \\ &\equiv R_{\ell,N}(x) \triangle U_{\ell,N}(x) \ .\end{aligned}$$

Hence $R_{\ell,N}(x) \triangle U_{\ell,N}(x)$ is meager in $E_{-\ell}(x)$ $\forall x \in D_{\ell,N}$. On the other hand,

$$I^{-1}(\{0, \ldots, \ell\}) = \bigcup_{N=0}^{\infty} (Q_{\ell,N} \times \{N\}) ,$$

where

$$Q_{\ell,N} = \{ x \in X \mid (E_{-\ell} \cap R_N)(x) \text{ is residual in } E_{-\ell}(x) \} .$$

Since

$$Q_{\ell,N} \cap D_{\ell,N} = \{ x \in D_{\ell,N} \mid (E_{-\ell} \cap U_N)(x) \text{ is dense in } E_{-\ell}(x) \} ,$$

it follows that $Q_{\ell,N}$ has the Baire property in X by Lemmas 2.7 and 2.9-(ii), which completes the proof of Claim 2.

By [6, Theorem 8.38], Claim 2, and Corollary 5.3, there is some dense G_δ subset $C_0 \subset X$ such that θ is continuous on C_0, I is continuous on $C_0 \times \mathbf{N}$, and $I(C_0 \times \mathbf{N}) \subset \mathbf{N}$.

For each $k \in \mathbf{Z}$, any non-empty open $U \subset X$ and all $x \in U$, let

$$\mathcal{Q}(x, U, k) = \bigcup_{i=0}^{\infty} (E_k \cap (U \times U))^i(x) .$$

The following properties are consequences of Definition 5.1-(iii)-(c),(d):

- for any $\epsilon > 0$, there is some $k \in \mathbf{Z}$ so that $\mathcal{Q}(x, U, k) \subset E_d^X(x, U, \epsilon)$ for all $x \in U$, and
- for every $k \in \mathbf{Z}$, there exists some $\epsilon > 0$ such that $E_d^X(x, U, \epsilon) \subset \mathcal{Q}(x, U, k)$ for all $x \in U$.

Hence, by hypothesis, there is some residual $C_1 \subset X$ such that, for any U, x and k as above, if $x \in C_1$, then $\mathcal{Q}(x, U, k)$ is somewhere dense. By Corollary 2.19-(ii), there is some residual $C \subset C_0 \cap C_1$ such that $E_k(x) \cap C$ is residual in $E_k(x)$ for all $x \in C$ and $k \in \mathbf{Z}$.

Fix $x, y \in C$ and some complete metric inducing the topology of Y

Claim 3. There exist sequences, (x_i) and (y_i) in C with $x_1 = x$ and $y_1 = y$, (g_i) and (h_i) in S_∞, (U_i) and (V_i) consisting of open subsets of X, and (n_i) and (k_i) in $\mathbf{N}$, such that:

(i) $g_i \cdot \theta(x) = \theta(x_i)$;

(ii) $h_i \cdot \theta(y) = \theta(y_i)$;

(iii) $x_{i+1} \in U_{i+1} \cap C \cap \mathcal{Q}(x_i, U_i, -n_i)$;

(iv) $y_{i+1} \in V_{i+1} \cap C \cap \mathcal{Q}(y_i, V_i, -k_i)$;

(v) $U_i \supset V_i \supset U_{i+1}$;

(vi) $\mathrm{diam}(\theta(U_i \cap C)) < 2^{-i}$;

(vii) $(U_{i+1} \cap C) \times \{N_{i+1}\} \subset I^{-1}(n_{i+1})$ for

$$N_{i+1} = \sup\{\, g_{i+1}(\ell), g_{i+1}^{-1}(\ell) \mid \ell \leq i+1 \,\} ;$$

(viii) $(V_{i+1} \cap C) \times \{K_i\} \subset I^{-1}(k_i)$ for

$$K_i = \sup\{\, h_i(\ell), h_i^{-1}(\ell) \mid \ell \leq i \,\} ;$$

(ix) $g_{j+1}(\ell) = g_{i+1}(\ell)$ and $g_{j+1}^{-1}(\ell) = g_{i+1}^{-1}(\ell)$ for $\ell \leq i+1 \leq j+1$;

(x) $h_j(\ell) = h_i(\ell)$ and $h_j^{-1}(\ell) = h_i^{-1}(\ell)$ for $\ell \leq i \leq j$;

(xi) $\mathcal{Q}(x_i, U_i, -n_i) \cap V_i$ is dense in V_i; and

(xii) $\mathcal{Q}(y_i, V_i, -k_i) \cap U_{i+1}$ is dense in U_{i+1}.

If this assertion is true, then there exist $g = \lim_i g_i$ and $h = \lim_i h_i$ in S_∞ by Claim 3-(ix),(x), and so $g \cdot \theta(x) = h \cdot \theta(y)$ by Claim 3-(i)–(vi), showing the result.

The construction of the sequences of Claim 3 is made by induction on $i \in \mathbf{N}$. Let $x_0 = x$, $U_0 = X$, $n_0 = 0$ and $g_0 = h_0 = 1_{S_\infty}$, and choose V_0 and k_0 so that $y \in V_0$ and

$$(V_0 \cap C) \times \{0\} \subset I^{-1}(k_0) .$$

Suppose that, for some fixed $i \in \mathbf{N}$, you have constructed all the terms of these sequences with indices $\leq i$. Then construct x_{i+1}, g_{i+1} and U_{i+1} in the following manner. (The construction of y_{i+1}, h_{i+1} and V_{i+1} is analogous.)

Take a non-empty open $U \subset V_i$ such that $\mathcal{Q}(y_i, V_i, -k_i) \cap U$ is dense in U. You may assume that $\operatorname{diam}(\theta(U \cap C)) < 2^{-i-1}$ because θ is continuous on C_0. Choose $x_{i+1} \in \mathcal{Q}(x_i, U_i, -n_i) \cap U$, and take $z_0, \ldots, z_k \in U_i$ so that $z_0 = x_i$, $z_k = x_{i+1}$ and $z_a \in E_{-n_i}(z_{a-1})$ for $a \in \{1, \ldots, k\}$. You may assume that $i > 0$ because (ix) does not restrict the choice of g_1.

Claim 4. We can assume that $z_a \in C$ for all $a \in \{0, \ldots, k\}$.

Claim 4 follows by showing the existence of elements

$$z'_a \in U_i \cap (E^{k-a}_{-n_i})^{-1}(P_U) \cap C$$

for $a \in \{0, \ldots, k\}$ so that $z'_0 = x_i$, and $z'_a \in E_{-n_i}(z'_{a-1})$ for $a \in \{1, \ldots, k\}$; then we can choose $x'_{i+1} = z'_k$ instead of x_{i+1}, and z'_a instead of z_a. We have

$$z'_0 = x_i \in U_i \cap (E^{k}_{-n_i})^{-1}(P_U) \cap C .$$

Now, assume that z'_a is constructed for some $a < k$. Since $z'_a \in C$ and $E^{k-a-1}_{-n_i}$ is continuous by Lemma 2.8-(i), the set

$$E_{n_i}(z'_a) \cap U_i \cap (E^{k-a-1}_{-n_i})^{-1}(P_U) \cap C$$

is residual in

$$E_{n_i}(z'_a) \cap U_i \cap (E^{k-a-1}_{-n_i})^{-1}(P_U) .$$

So, by Remark 9-(ii), there is some

$$z'_{a+1} \in E_{n_i}(z'_a) \cap U_i \cap (E^{k-a-1}_{-n_i})^{-1}(P_U) \cap C$$

as desired.

Continuing with the proof of Claim 3, Claim 4 gives $I(z_a, N_i) = n_i$ for all $a \in \{0, \ldots, k\}$ by the induction hypothesis with Claim 3-(vii).

Claim 5. We can assume that, for each $a < k$, there exists some $f_a \in W_{N_i}$ such that $f_a \cdot \theta(z_a) = \theta(z_{a+1})$.

Like in Claim 4, we show that the condition of this claim is satisfied by a new finite sequence of points

$$z'_a \in U_i \cap (E^{k-a}_{-n_i})^{-1}(P_U) \cap C$$

so that $z'_0 = x_i$ and $z'_a \in E_{-n_i}(z'_{a-1})$ for $a \in \{1, \ldots, k\}$; in particular, $I(z'_a, N_i) = n_i$ as above. This new sequence is constructed by induction on a. First, let $z'_0 = x_i$.

Now, assume that z'_a was constructed for some $a < k$. Since $I(z'_a, N_i) = n_i$, $\forall^* z \in E_{-n_i}(z'_a)$, $\exists f \in W_{N_i}$ so that $f \cdot \theta(z_a) = \theta(z)$. So the set of points

$$z \in E_{-n_i}(z'_a) \cap U_i \cap (E_{-n_i}^{k-a-1})^{-1}(P_U) \cap C$$

such that $\exists f \in W_{N_i}$ so that $f \cdot \theta(z_a) = \theta(z)$ is residual in

$$E_{-n_i}(z'_a) \cap U_i \cap (E_{-n_i}^{k-a-1})^{-1}(P_U) \cap C \ .$$

Hence $f_a \cdot \theta(z'_a) = \theta(z'_{a+1})$ for some $f_a \in W_{N_i}$ and some

$$z'_{a+1} \in E_{-n_i}(z'_a) \cap U_i \cap (E_{-n_i}^{k-a-1})^{-1}(P_U) \cap C$$

by Remark 9-(ii), completing the proof of Claim 5.

According to Claim 5, $f_i^* \cdot \theta(x_i) = \theta(x_{i+1})$ for $f_i^* = f_{k-1} \cdots f_0 \in W_{N_i}$. Then let $g_{i+1} = f_i^* g_i$. Moreover we can take some open neighborhood U_{i+1} of x_{i+1} in U and some $n_{i+1} \in \mathbf{N}$ such that $\operatorname{diam}(\theta(U_{i+1} \cap C)) < 2^{-i-1}$ and

$$(U_{i+1} \cap C) \times \{N_{i+1}\} \subset I^{-1}(n_{i+1}) \ ,$$

where N_{i+1} is defined according Claim 3-(vii). These choices of x_{i+1}, g_{i+1}, U_{i+1} and n_{i+1} satisfy the conditions of Claim 3. □

Remark 10. This proof is inspired by that of [5, Theorem 3.18].

6 A class of turbulent metric relations

Let X be a set. Consider a family of relations, $\mathcal{U} = \{U_{R,r} \subset X \times X \mid R, r > 0\}$, over X satisfying the following hypothesis.

Hypothesis 1. (i) $\bigcap_{R,r>0} U_{R,r} = \Delta_X$;

(ii) each $U_{R,r}$ is symmetric;

(iii) if $R \leq S$, then $U_{R,r} \supset U_{S,r}$ for all $r > 0$;

(iv) $U_{R,r} = \bigcup_{s<r} U_{R,s}$ for all $R, r > 0$; and

(v) there is some function $\phi : (\mathbf{R}_+)^2 \to \mathbf{R}_+$ such that, for all $R, S, r, s > 0$,

$$\begin{gathered} R \leq \phi(R, r) \ , \\ (R \leq S \ \& \ r \leq s) \Longrightarrow \phi(R, r) \leq \phi(S, s) \ , \\ U_{\phi(R,r+s),r} \circ U_{\phi(R,r+s),s} \subset U_{R,r+s} \ . \end{gathered}$$

By Hypothesis 1, the sets $U_{R,r}$ form a base of entourages of a Hausdorff uniformity, also denoted by $\mathcal{U}$, on X. This uniformity is metrizable because the entourages $U_{n,1/n}$, $n \in \mathbf{Z}_+$, form a countable base for it.

For each $r > 0$, let $E_r = \bigcap_{R>0} U_{R,r}$. This set is symmetric by Hypothesis 1-(ii); moreover

$$E_s \circ E_r \subset E_{r+s} \ , \tag{14}$$

for $r, s > 0$, by Hypothesis 1-(v).

Lemma 6.1. *For $R, r > 0$ and $S = \phi(\phi(R, r), r)$ (where ϕ is the function given in Hypothesis 1-*(v)*), the set $U_{S,r} \subset \mathrm{Int}(U_{R,r})$.*

Proof. Let $(x, y) \in U_{S,r}$. By Hypothesis 1-(iv), there is some $r_0 < r$ such that $(x, y) \in U_{S,r_0}$. Let $r_1 = \dfrac{r - r_0}{2}$. By Hypothesis 1-(v),

$$
\begin{aligned}
&U_{S,r_1} \circ U_{S,r_0} \circ U_{S,r_1} \\
&\qquad \subset U_{\phi(\phi(R,r),\frac{r+r_0}{2}),r_1} \circ U_{\phi(\phi(R,r),\frac{r+r_0}{2}),r_0} \circ U_{\phi(R,r),r_1} \\
&\qquad\qquad \subset U_{\phi(R,r),\frac{r+r_0}{2}} \circ U_{\phi(R,r),r_1} \subset U_{R,r} \ .
\end{aligned}
$$

So, by Hypothesis 1-(ii), $U_{S,r_1}(x) \times U_{S,r_1}(y) \subset U_{R,r}$, which implies that $(x, y) \in \mathrm{Int}(U_{R,r})$. □

Corollary 6.2. *For each $r > 0$, the set $E_r = \bigcap_{R>0} \mathrm{Int}(U_{R,r})$.*

Hypothesis 1-(iii) and Corollary 6.2 imply that $E_r = \bigcap_{n=1}^{\infty} \mathrm{Int}(U_{n,r})$ for all $r > 0$ and so E_r is a G_δ subset of $X \times X$. Hence the relations E_r satisfy Definition 5.1-(iii)-(a),(b).

Let $d : X \times X \to [0, \infty]$ be defined by

$$
d(x, y) = \inf\{\, r > 0 \mid (x, y) \in E_r \,\} \ ; \tag{15}
$$

in particular, $d(x, y) = \infty$ if x is not in any of $E_r(y)$, $r > 0$. It easily follows from Hypothesis 1 that d is a metric relation over X. Observe also that

$$
B_d(x, r) \subset E_r(x) \subset B_d(x, s)
$$

for $0 < r < s$. Therefore

$$
E_d^X = \bigcup_{r>0} E_r \ , \tag{16}
$$

and $B_d(x, r) \subset U_{R,r}(x)$ for all $R, r > 0$ and $x \in X$, which implies that the topology induced by d on X is finer than the topology induced by the uniformity $\mathcal{U}$ on X. Consequently, d satisfies the conditions (ii) and (iii)-(c),(d) of Definition 5.1 with the relations E_r.

Example 6.3. Let $\{d_R \mid R > 0\}$ be a family of pseudo-metrics on a set, X, such that

$$
R \le S \Longrightarrow d_R \le d_S \ , \tag{17}
$$

$$
(\, d_R(x, y) = 0 \ \forall R > 0 \,) \Longrightarrow x = y \ . \tag{18}
$$

Then the sets

$$
U_{R,r} = \{\, (x, y) \in X \times X \mid d_R(x, y) < r \,\}
$$

clearly satisfy Hypothesis 1; in particular, Hypothesis 1-(v) is satisfied with $\phi(R, r) = R$ since the triangle inequality of each d_R and (17) give

$$
U_{R,r} \circ U_{S,s} \subset U_{\min\{R,S\},r+s} \tag{19}
$$

for all $R, S, r, s > 0$. It follows that $U_{R,r}(x)$ is open for all $x \in X$ and $R, r > 0$. In this case, the relations $U_{R,r}$ induce the topology defined by the family of pseudo-metrics d_R, and the corresponding sets E_r define the metric relation $d = \sup_{R>0} d_R$.

To prove that d, the metric equivalence relation given by (15), satisfies the remaining conditions of Definition 5.1, suppose that the following additional requirement is satisfied.

Hypothesis 2. (i) X is a Polish space (with the topology induced by the uniformity $\mathcal{U}$);

(ii) for all $R, r, s > 0$, for all $x \in X$, if $y \in E_s(x)$, then there are some $T, t > 0$ such that $U_{T,t}(y) \subset E_s \circ U_{R,r}(x)$; and

(iii) for all $r, s > 0$, for all $x \in X$, if $y \in E_s(x)$ and V is a neighborhood of y in X, then there is a neighborhood W of y in X such that

$$E_r(W) \cap E_r(E_s(x)) \subset E_r(V \cap E_s(x)) \ .$$

Proposition 6.4. *If $\mathcal{U}$ satisfies Hypothesis 2, then d is of type I.*

Proof. It only remains to show that d satisfies Definition 5.1-(iii)-(e) and (f).

Hypothesis 2-(ii) simply means that E_s is open and hence continuous because it is symmetric.

Let $r, s, t > 0$, $x \in X$ and $y \in E_s(x)$. Suppose that $E_r \circ E_s \supset E_t$, and let V be a neighborhood of y in X. By Hypothesis 2-(iii), there is some open neighborhood W of y in X such that

$$E_r(W) \cap E_t(x) \subset E_r(W) \cap E_r(E_s(x)) \subset E_r(V \cap E_s(x)).$$

Since $E_r(W)$ is open in X, this proves that $E_r \cap (E_s(x) \times E_t(x))$ is an open relation over $E_s(x)$ and $E_t(x)$. □

Remark 11. In some applications, the following condition, which is stronger than Hypothesis 2-(ii), is satisfied: for all $R, r, s > 0$, there are some $T, t > 0$ such that $U_{T,t} \circ E_s \subset E_s \circ U_{R,r}$. This means that each E_s is "uniformly open" (or "uniformly continuous," because it is symmetric).

To show that d is turbulent, assume also the following additional hypothesis.

Hypothesis 3. (i) E_d^X has more than one equivalence class;

(ii) for any $x, y \in X$ and any $R, r > 0$, there is some $s > 0$ such that $U_{R,r}(x) \cap E_s(y) \neq \emptyset$; and

(iii) for any $R, r > 0$ and any $x \in X$, there are some $S, s > 0$, some dense subset $\mathcal{D} \subset U_{S,s}(x) \cap E_d^X(x)$, and some d-dense subset of $\mathcal{D}$ whose points can be joined by d-continuous paths in $U_{R,r}(x)$.

Lemma 6.5. *The relation E_d^X is minimal.*

Proof. This follows from Hypothesis 3-(ii) and (16). □

Lemma 6.6. *If $r < s$, then $\overline{E_r(x)} \subset E_s(x)$ for all $x \in X$.*

Proof. If $y \in \overline{E_r(x)}$ and $R > 0$, then $U_{\phi(R,s),s-r}(y) \cap U_{\phi(R,s),r}(x) \neq \emptyset$. So $y \in \bigcap_{R>0} U_{R,s} = E_s(x)$ by Hypothesis 1-(ii),(v). □

Lemma 6.7. $\mathrm{Int}(E_r(x)) = \emptyset$ *for all* $x \in X$ *and* $r > 0$.

Proof. Suppose that $\mathrm{Int}(E_r(x)) \neq \emptyset$. Then, for each $y \in X$, the intersection $E_s(y) \cap E_r(x) \neq \emptyset$ for some $s > 0$, by Lemma 6.5 and (16). Therefore $y \in E_{r+s}(x)$ by (14). It follows that $X = E_d^X(x)$ by (16), contradicting Hypothesis 3-(i). □

Proposition 6.8. *The relation* E_d^X *is turbulent.*

Proof. The relation E_d^X is minimal because of Lemma 6.5. Each equivalence class of E_d^X is meager because of Lemmas 6.6 and 6.7 and (16). Finally, the local equivalence classes of E_d^X are somewhere dense because of Hypothesis 3-(iii). □

Theorem 5.4, and Propositions 6.4 and 6.8 have the following immediate consequence.

Proposition 6.9. *For any Polish* S_∞*-space* Y*, the relation* E_d^X *is generically* $E_{S_\infty}^Y$*-ergodic.*

Remark 12. It is easy to somewhat weaken Hypothethis 3-(ii) to treat generic turbulence as we will do in a subsequent work.

Assume that the following final hypothesis (Hypothesis 4) is satisfied. This hypothesis will be used to obtain that Proposition 6.9 does not follow from the results of [5] on Polish actions.

Hypothesis 4. For all $r > 0$ and residually many $x, y \in X$, there exists $s_0 > 0$ such that $E_s(y) \setminus E_r(x)$ is dense in $E_s(y)$ for all $s > s_0$.

Proposition 6.10. *The relation* $E_d^X \not\leq_B E_G^Y$*, for any Polish group* G *and any Polish* G*-space* Y.

Proof. Suppose that, for some Polish group G and some Polish G-space Y, there exists an (E_d^X, E_G^Y)-invariant Borel map $\theta : X \to Y$ such that $\bar{\theta} : X/E_d^X \to G\backslash X$ is injective. Fix complete metrics d_X on X and d_G on G. The following claim holds because of [6, Theorem 8.38], Lemmas 5.2 and 6.6, Corollary 2.19-(ii) and Hypothesis 4.

Claim 6. There is a residual subset $C \subset X$ such that:

(i) $\theta : C \to Y$ is continuous;

(ii) for any neighborhood W of 1_G in G, $\forall n \in \mathbf{Z}_+$, $\forall x \in C$ and $\forall^* x' \in E_n(x)$, $\exists \epsilon > 0$ such that, $\forall^* x'' \in B_{d_X}(x, \epsilon) \cap E_n(x')$, $\exists h \in W$ so that $h \cdot \theta(x) = \theta(x'')$;

(iii) $E_n(x) \cap C$ is residual in $E_n(x)$ $\forall x \in C$ and $\forall n \in \mathbf{Z}_+$; and

(iv) $\forall r > 0$ and $\forall x, y \in C$, $\exists s_0 > r$ such that, $\forall s > s_0$, $E_s(y) \setminus E_r(x)$ is residual in $E_s(y)$.

It may be further assumed that $C = \bigcap_{i=0}^\infty O_i$, where each O_i is a dense open subset of X. Let $\phi : (\mathbf{R}_+)^2 \to \mathbf{R}_+$ be the function given by Hypothesis 1-(v).

Claim 7. There are sequences (x_i) and (x'_i) in X, a sequence (U_i) of open subsets of X, a sequence (g_i) in G, and sequences (n_i), (n'_i), (R_i) and (R'_i) in $\mathbf{Z}_+$, and (r_i) in $\mathbf{R}_+$, such that:

(i) $\sum_{i=0}^{\infty} r_i < 1$;

(ii) $n_i < n'_i < n_{i+1}$ and $m_i := n_i - 2\sum_{j=0}^{i-1} n'_j \uparrow \infty$ (in particular, $m_0 = n_0$);

(iii) $R_i \leq R'_i$ and $\phi(\phi(\phi(\phi(R'_i, n_{i-1}), m_{i-1}), 2), 1) \leq R_{i+1}$ (with $n_{-1} = m_{-1} = 1$);

(iv) $x_i \in U_i \cap C$;

(v) $x'_i \in E_{n'_i}(x_i) \cap C$;

(vi) $x_{i+1} \notin U_{R'_{i+1},n_i}(x_i)$;

(vii) $x_{i+1} \in U_{R_{i+1},r_{i+1}}(x_i) \cap E_{n'_i}(x'_i)$;

(viii) $U_{R_{i+1},r_{i+1}}(x_i) \subset B_{d_X}(x_i, 2^{-i})$;

(ix) $\overline{U_{i+1}} \subset U_i \cap O_{i+1}$;

(x) $g_i \cdot \theta(x_0) = \theta(x_i)$; and

(xi) $d_G(g_{i+1}, g_i) < 2^{-i}$.

The sequences of this assertion are constructed by induction on i. Because C is residual in X, you can choose any $x_0 \in C$, any open neighborhood U_0 of x_0 in X, $g_0 = 1_G$, any $R_0, R'_0, n_0 \in \mathbf{Z}_+$ with $R_0 \leq R'_0$, and $0 < r_0 < 1/2$.

Suppose that, for some $i \in \mathbf{N}$, you have defined g_j, U_j, x_j, n_i, R_j, R'_j and r_j for $j \leq i$, and n'_j and x'_j for $j < i$. Fix any open neighborhood W of 1_G in G such that $d_G(g_i, gg_i) < 2^{-i}$ for all $g \in W$. By Claims 6-(iv) and 7-(iv), there is some integer $n'_i > n_i$ such that $E_{n'_i}(y) \setminus E_{n_i}(x_i)$ is residual in $E_{n'_i}(y)$ $\forall y \in C$. By Claims 7-(iv) and 6-(ii),(iii), and since $E_{n'_i}(x_i)$ is a Baire space (Remark 9-(ii)), there is some $x'_i \in E_{n'_i}(x_i) \cap C$ such that $E_{n'_i}(x'_i) \cap C$ is residual in $E_{n'_i}(x'_i)$, and there is some $\epsilon > 0$ so that, $\forall^* x'' \in B_{d_X}(x_i, \epsilon) \cap E_{n'_i}(x'_i)$, $\exists g \in W$ such that $g \cdot \theta(x_i) = \theta(x'')$. Then choose $R_{i+1} \geq \phi(\phi(\phi(\phi(R'_i, n_{i-1}), m_{i-1}), 2), 1)$ and $0 < r_{i+1} < 2^{-i-2}$ so that

$$U_{R_{i+1},r_{i+1}}(x_i) \subset B_{d_X}(x_i, \min\{\epsilon, 2^{-i}\}) \cap U_i \ .$$

Since $E_{n'_i}(x'_i) \setminus E_{n_i}(x_i)$ and $E_{n'_i}(x'_i) \cap C$ are residual in $E_{n'_i}(x'_i)$, the set

$$(\mathrm{Int}(U_{R_{i+1},r_{i+1}}(x_i)) \cap E_{n'_i}(x'_i) \cap C) \setminus E_{n_i}(x_i)$$

is residual in $\mathrm{Int}(U_{R_{i+1},r_{i+1}}(x_i)) \cap E_{n'_i}(x'_i)$, and therefore it is non-empty since $E_{n'_i}(x'_i)$ is a Baire space; thus we can choose x_{i+1} in this set. Since $x_{i+1} \notin E_{n_i}(x_i)$, there is some $R'_{i+1} > R_{i+1}$ such that $x_{i+1} \notin U_{R'_{i+1},n_i}(x_i)$. Because $x_{i+1} \in B_{d_X}(x_i, \epsilon) \cap E_{n'_i}(x'_i)$, there is some $g \in W$ such that $g \cdot \theta(x_i) = x_{i+1}$. With $g_{i+1} = gg_i$, we get

$$g_{i+1} \cdot \theta(x_0) = gg_i \cdot \theta(x_0) = g \cdot \theta(x_i) = \theta(x_{i+1}) \ .$$

Choose $n_{i+1} > n'_i$ such that $m_{i+1} > m_i$. Finally, we can take an open neighborhood U_{i+1} of x_{i+1} in X so that $\overline{U_{i+1}} \subset U_i \cap O_{i+1}$ because $x_{i+1} \in U_i \cap O_{i+1}$, completing the proof of Claim 7.

From Claim 7-(vii),(viii),(xi), it follows that $\exists \lim_i x_i = x_\infty$ in X and $\exists \lim_i g_i = g_\infty$ in G.

Claim 8. $x_{i+1} \notin U_{\phi(R'_{i+1},n_i),m_i}(x_0)$ for all $i \in \mathbf{N}$.

It follows from Claim 7-(v),(vii) and (14), that

$$x_{j+1} \in \big(E_{n'_j} \circ E_{n'_j}\big)(x_j) \subset E_{2n'_j}(x_j)$$

for all $j \in \mathbf{N}$, and so, because of (14), that

$$x_i \in \big(E_{2n'_{i-1}} \circ \cdots \circ E_{2n'_0}\big)(x_0) \subset E_{2\sum_{j=0}^{i-1} n'_j}(x_0)$$

for all $i \in \mathbf{N}$. If $x_{i+1} \in U_{\phi(R'_{i+1},n_i),m_i}(x_0)$, then

$$x_{i+1} \in \big(U_{\phi(R'_{i+1},n_i),m_i} \circ E_{2\sum_{j=0}^{i-1} n'_j}\big)(x_i) \subset U_{R'_{i+1},n_i}(x_i)$$

by Hypothesis 1-(ii),(v), which contradicts Claim 7-(vi). Thus Claim 8 is justified.

For any fixed $i \in \mathbf{N}$ with $m_i > 2$, take some integer $k \geq i+3$ so that $x_k \in U_{\phi(\phi(\phi(R'_{i+1},n_i),m_i),2),1}(x_\infty)$.

Claim 9. $x_k \in U_{R'_\ell,\sum_{j=\ell+1}^k r_j}(x_\ell)$ for all $\ell \in \{0,\ldots,k-1\}$.

This claim is proved by backwards induction on ℓ. For $\ell = k-1$ the assertion follows from Claim 7-(iii),(vii).

Suppose now that Claim 9 holds for some $\ell \in \{1,\ldots,k-2\}$. Then

$$\begin{aligned}
x_k &\in \big(U_{R'_\ell,\sum_{j=\ell+1}^k r_j} \circ U_{R_\ell,r_\ell}\big)(x_{\ell-1}) \\
&\subset \big(U_{\phi(R'_{\ell-1},\sum_{j=\ell}^k r_j),\sum_{j=\ell+1}^k r_j} \circ U_{\phi(R'_{\ell-1},\sum_{j=\ell}^k r_j),r_\ell}\big)(x_{\ell-1}) \\
&\subset U_{R'_{\ell-1},\sum_{j=\ell}^k r_j}(x_{\ell-1})
\end{aligned}$$

by Claim 7-(i)–(iii), (vii) and Hypothesis 1-(v), because $\sum_{j=\ell}^k r_j < 1 = n_{-1} \leq n_{\ell-2}$.

Claim 10. $x_\infty \notin U_{\phi(\phi(R'_{i+1},n_i),m_i),m_i-2}(x_0)$.

By Claims 7-(i),(iii),(vii) and 9, and Hypothesis 1-(v),

$$\begin{aligned}
x_k &\in \big(U_{R'_{i+2},\sum_{j=i+3}^k r_j} \circ U_{R'_{i+2},r_{i+2}}\big)(x_{i+1}) \\
&\subset \big(U_{\phi(\phi(\phi(\phi(R'_{i+1},n_{i-1}),m_{i-1}),2),1),\sum_{j=i+3}^k r_j} \\
&\quad \circ U_{\phi(\phi(\phi(\phi(R'_{i+1},n_{i-1}),m_{i-1}),2),1),r_{i+2}}\big)(x_{i+1}) \\
&\subset U_{\phi(\phi(\phi(R'_{i+1},n_i),m_i),2),1}(x_{i+1})\ .
\end{aligned}$$

So, if $x_\infty \in U_{\phi(\phi(R'_{i+1},n_i),m_i),m_i-2}(x_0)$, then

$$\begin{aligned}
x_{i+1} &\in \big(U_{\phi(\phi(\phi(R'_{i+1},n_i),m_i),2),1} \circ U_{\phi(\phi(\phi(R'_{i+1},n_i),m_i),2),1} \\
&\quad \circ U_{\phi(\phi(R'_{i+1},n_i),m_i),m_i-2}\big)(x_0) \\
&\subset \big(U_{\phi(\phi(R'_{i+1},n_i),m_i),2} \circ U_{\phi(\phi(R'_{i+1},n_i),m_i),m_i-2}\big)(x_0) \\
&\subset U_{\phi(R'_{i+1},n_i),m_i}(x_0)
\end{aligned}$$

by Hypothesis 1-(ii),(v), which contradicts Claim 8, and so Claim 10 is justified.

From Claims 10 and 7-(i),(ii), it follows that $x_\infty \notin E_d^X(x_0)$. On the other hand, $x_\infty \in C$ by Claim 7-(iv),(ix). Hence

$$g_\infty \cdot \theta(x_0) = \lim_i g_i \cdot \theta(x_0) = \lim_i \theta(x_i) = \theta(\lim_i x_i) = \theta(x_\infty)$$

by Claims 6-(i) and 7-(iv),(x). So $(\theta(x_0), \theta(x_\infty)) \in E_G^X$, giving $(x_0, x_\infty) \in E_d^X$, which is a contradiction. □

Remark 13. The proof of Proposition 6.10 is inspired by that of [5, Theorem 8.2], which however does not seem to fit into our conditions.

7 The supremum metric relation

A concrete case of Example 6.3 is $C(\mathbf{R})$, the space of real valued continuous functions on $\mathbf{R}$ endowed with the compact-open topology, and the supremum metric relation, d_∞, which is induced by the supremum norm, $\| \ \|_\infty$, defined by $\|f\|_\infty = \sup_{x\in\mathbf{R}} |f(x)|$. For each $R > 0$, let d_R be the pseudo-metric on $C(\mathbf{R})$ induced by the semi-norm $\| \ \|_R$ given by by $\|f\|_R = \sup_{|x|<R} |f(x)|$. Clearly, this family of pseudo-metrics satisfies the conditions (17) and (18), and induces the compact-open topology of $C(\mathbf{R})$. Moreover $d_\infty = \sup_{R>0} d_R$. In this case, each $U_{R,r}$ (respectively, E_r) consists of the pairs (f, g) that satisfy $\|f - g\|_R < r$ (respectively, $|f(x) - g(x)| < r$ for all $x \in \mathbf{R}$).

The following notation will be used: $E_\infty = E_{d_\infty}^{C(\mathbf{R})}$, and $B_\infty(f, r) = B_{d_\infty}(f, r)$ for each $f \in C(\mathbf{R})$ and $r > 0$. Two functions, $f, g \in C(\mathbf{R})$, are in the same equivalence class of E_∞ if and only if $f - g$ is bounded; in particular, the bounded functions of $C(\mathbf{R})$ form an equivalence class of E_∞.

The following theorem follows from Propositions 6.4 and 6.8–6.10 once Hypotheses 1–4 are shown in this case.

Theorem 7.1. *The following are true:*

(*i*) d_∞ *is turbulent;*

(*ii*) E_∞ *is generically* $E_{S_\infty}^Y$*-ergodic for any Polish* S_∞*-space* Y*; and*

(*iii*) $E_\infty \not\leq_B E_G^Y$ *for any Polish group* G *and any Polish* G*-space* Y*.*

Remark 14. Let $C_b(\mathbf{R}) \subset C(\mathbf{R})$ be the subset of bounded continuous functions. The sum of functions makes the space $C(\mathbf{R})$ into a Polish group, and $C_b(\mathbf{R})$ into a subgroup. The orbit relation of the action of $C_b(\mathbf{R})$ on $C(\mathbf{R})$ given by translation is E_∞. Therefore, by virtue of Theorem 7.1-(iii), there is no Polish topology on $C_b(\mathbf{R})$ with respect to which this action is continuous.

For instance, consider the restriction of the compact-open topology to $C_b(\mathbf{R})$. Then the action of $C_b(\mathbf{R})$ on $C(\mathbf{R})$ is continuous, $C_b(\mathbf{R})$ is metrizable because $C(\mathbf{R})$ is completely metrizable, and $C_b(\mathbf{R})$ is separable because it contains $C_0(\mathbf{R})$, which is dense in $C(\mathbf{R})$ and separable (by the Stone-Weierstrass theorem). But $C_b(\mathbf{R})$ is not

completely metrizable with the compact-open topology; in particular, it is not closed in $C(\mathbf{R})$.

Consider now the topology on $C_b(\mathbf{R})$ induced by $\| \|_\infty$. Then the action of $C_b(\mathbf{R})$ on $C(\mathbf{R})$ is continuous, and $C_b(\mathbf{R})$ is completely metrizable; indeed, it is a Banach algebra with $\| \|_\infty$. However $C_b(\mathbf{R})$ is not separable with $\| \|_\infty$, which can be shown as follows. For each $x \in \{\pm 1\}^{\mathbf{Z}}$, let $\tilde{x} \in C_b(\mathbf{R})$ be the function whose graph is the union of segments between all consecutive points in the graph of x. Then $\{ B_\infty(\tilde{x}, 1) \mid x \in \{\pm 1\}^{\mathbf{Z}} \}$ is an uncountable family of disjoint open subsets of $C_b(\mathbf{R})$. So $C_b(\mathbf{R})$ is not second countable, and therefore it is not separable.

According to Example 6.3, the sets $U_{R,r}$ satisfy Hypothesis 1 and induce d_∞. In this case, the inclusion (14) becomes the equality

$$E_r \circ E_s = E_{r+s} \tag{20}$$

for all $r, s > 0$; this holds because, if $g \in E_{r+s}(f)$, then

$$f + \frac{s}{r+s}(g - f) \in E_r(g) \cap E_s(f) \; .$$

It is well known that $C(\mathbf{R})$ is Polish (Hypothesis 2-(i)). The following lemma shows that Hypothesis 2-(ii) is satisfied in this case.

Lemma 7.2. $U_{R,r} \circ E_s = E_s \circ U_{R,r} = U_{R,r+s}$ *for all* $R, r, s > 0$.

Proof. If $S \geq R$, then

$$d_R(f,h) \leq d_R(f,g) + d_R(g,h) \leq d_R(f,g) + d_S(g,h)$$

for all $f, g, h \in C(\mathbf{R})$, because $d_R \leq d_S$. This implies that $U_{R,r} \circ U_{S,s}$ and $U_{S,s} \circ U_{R,r}$ are both contained in $U_{R,r+s}$, which in turn implies that $U_{R,r} \circ E_s$ and $E_s \circ U_{R,r}$ are both contained in $U_{R,r+s}$.

To prove the reverse inclusions, let $f \in C(\mathbf{R})$ and $g \in U_{R,r+s}(f)$. Then

$$h_0 = f + \frac{s}{r+s}(g - f) \in U_{R,s}(f) \cap U_{R,r}(g) \; ,$$
$$h_1 = f + \frac{r}{r+s}(g - f) \in U_{R,r}(f) \cap U_{R,s}(g) \; .$$

By continuity, $h_0 \in U_{S,s}(f)$ and $h_1 \in U_{S,s}(g)$ for some $S > R$. Let $\lambda : \mathbf{R} \to [0,1]$ be any continuous function supported in $[-S, S]$ such that $\lambda \equiv 1$ on $[-R, R]$. Then

$$f + \lambda(h_0 - f) \in E_s(f) \cap U_{R,r}(g) \; ,$$
$$g + \lambda(h_1 - g) \in U_{R,r}(f) \cap E_s(g) \; ,$$

which implies that $g \in (U_{R,r} \circ E_s)(f) \cap (E_s \circ U_{R,r})(f)$. $\square$

Corollary 7.3. *If* $R, S, r, s > 0$, *then* $U_{R,r} \circ U_{S,s} = U_{\min\{R,S\},r+s}$.

Proof. The inclusion "$\subset$" is (19), and the inclusion "$\supset$" follows from Lemma 7.2. $\square$

Because (20) and Lemma 7.2, and because the sets $U_{R,r}$ are open in Example 6.3, the following lemma implies Hypothesis 2-(iii) in this case.

Lemma 7.4. *If $T, r, s, t > 0$, $f \in C(\mathbf{R})$ and $g \in E_s(f)$ are such that $U_{T,t'}(g) \subset U_{T,s}(f)$ for some $t' > t$, then*

$$U_{T,t+r}(g) \cap E_{r+s}(f) = E_r(U_{T,t}(g) \cap E_s(f)) \ .$$

Proof. The inclusion "$\supset$" follows from (14) and Lemma 7.2. To prove "$\subset$", let $h \in U_{T,t+r}(g) \cap E_{r+s}(f)$. By (20) and Lemma 7.2, there are some $g_0 \in E_r(h) \cap U_{T,t}(g)$ and $f_0 \in E_r(h) \cap E_s(f)$. By continuity, $g_0 \in U_{T',t}(g) \subset U_{T',s}(f)$ for some $T' > T$. Let $\lambda : \mathbf{R} \to [0,1]$ be any continuous function supported in $[-T', T']$ such that $\lambda \equiv 1$ on $[-T, T]$. Then

$$f_0 + \lambda(g_0 - f_0) \in E_r(h) \cap U_{T,t}(g) \cap E_s(f) \ ,$$

obtaining that $h \in E_r(U_{T,t}(g) \cap E_s(f))$. □

The fact that E_∞ has more than one class (Hypothesis 3-(i)) is obvious because $d_\infty(f, g) = \infty$ if f is bounded and g unbounded. Hypotheses 3-(ii),(iii) and 4 are a consequence of the following lemmas.

Lemma 7.5. *For every $f, g \in C(\mathbf{R})$ and every $R, r > 0$, if $s > d_{R'}(f, g)$ for some $R' > R$, then $U_{R,r}(f) \cap E_s(g) \neq \emptyset$.*

Proof. Let $\lambda : \mathbf{R} \to [0,1]$ be a continuous function supported in $[-R', R']$ such that $\lambda \equiv 1$ on $[-R, R]$. Then $g + \lambda(f - g) \in U_{R,r}(f) \cap E_s(g)$. □

Lemma 7.6. *For every $R, r > 0$ and every $f \in C(\mathbf{R})$, the set $U_{R,r}(f) \cap E_\infty(f)$ is d_∞-path connected.*

Proof. For every $g \in U_{R,r}(f) \cap E_\infty(f)$, the mapping $t \mapsto tf + (1-t)g$ defines a d_∞-continuous path in $U_{R,r}(f) \cap E_\infty(f)$ from g to f. □

Lemma 7.7. *If $0 < r < s$, then $E_s(g) \setminus E_r(f)$ is dense in $E_s(g)$ for all $f, g \in C(\mathbf{R})$.*

Proof. It has to be shown that if $T, t > 0$ and $h \in E_s(g)$, then $E_s(g) \setminus E_r(f)$ and $U_{T,t}(h)$ have non-empty intersection. Let $T'' > T' > T$ and $0 < 2\epsilon < s - r$, and let $\lambda : \mathbf{R} \to [0,1]$ be a continuous function with support in $[-T', T']$ and such that $\lambda \equiv 1$ on $[-T, T]$. Because f and g are uniformly continuous on $[T', T'']$, there is some $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ and $|g(x) - g(y)| < \epsilon$ if $|x - y| < \delta$ for all $x, y \in [T', T'']$. Take any continuous function $\mu : \mathbf{R} \to (-s, s)$, supported in $[T', T'']$, such that $|\mu(x_0) - \mu(y_0)| > 2s - \epsilon$ for some $x_0, y_0 \in (T', T'')$ with $|x_0 - y_0| < \delta$. Then

$$u = g + \lambda(h - g) + \mu \in E_s(g) \cap U_{T,t}(h) \ .$$

Moreover $u \notin E_r(f)$, otherwise we get the following contradiction:

$$\begin{aligned} \epsilon > |g(x_0) - g(y_0)| &= |(u - \mu)(x_0) - (u - \mu)(y_0)| \\ &\geq |\mu(x_0) - \mu(y_0)| - |u(x_0) - f(x_0)| - |f(x_0) - f(y_0)| - |f(y_0) - u(y_0)| \\ &> 2(s - r - \epsilon) > 2\epsilon \ . \end{aligned}$$ □

Remark 15. The symmetric relations over $C(\mathbf{R})$ with fibers the balls $B_\infty(f,r)$ cannot be used instead of the relations E_r to show that d_∞ is of type I. For instance, each ball $B_\infty(f,r)$ is not G_δ in $C(\mathbf{R})$; otherwise it would be Polish, and therefore it would be a Baire space with the induced topology. But $\emptyset$ is residual in $B_\infty(f,r)$ for all $r>0$, as the following argument shows. Take sequences $0<r_n\uparrow r$ and $0<R_n\uparrow\infty$. For each n, let U_n be the set of functions $g\in B_\infty(f,r)$ such that

$$\sup_{|x|>R_n}|f(x)-g(x)|>r_n\ .$$

It is easy to check that the sets U_n are open and dense in $B_\infty(f,r)$ and their intersection is empty.

8 The Gromov space

Let M be a metric space and let d_M, or simply d, be its distance function. The *Hausdorff distance* between two subsets, $A,B\subset M$, is given by

$$H_d(A,B)=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\right\}\ .$$

Observe that $H_d(A,B)=H_d(\overline{A},\overline{B})$, and $H_d(A,B)=0$ if and only if $\overline{A}=\overline{B}$. Also, it is well known and easy to prove that H_d satisfies the triangle inequality, and its restriction to the family of compact subsets of M is finite valued, and moreover complete if M is complete.

Let M and N be arbitrary metric spaces. A metric on $M\sqcup N$ is called *admissible* if its restrictions to M and N are d_M and d_N, where M and N are identified with their canonical injections in $M\sqcup N$. The *Gromov-Hausdorff distance* (or *GH distance*) between M and N is defined by

$$d_{GH}(M,N)=\inf_d H_d(M,N)\ ,$$

where the infimum is taken over all admissible metrics d on $M\sqcup N$. It is well known that $d_{GH}(M,N)=d_{GH}(\overline{M},\overline{N})$, where $\overline{M}$ and $\overline{N}$ denote the completions of M and N, $d_{GH}(M,N)=0$ if and only if $\overline{M}$ and $\overline{N}$ are isometric, d_{GH} satisfies the triangle inequality, and $d_{GH}(M,N)<\infty$ if $\overline{M}$ and $\overline{N}$ are compact.

There is also a pointed version of d_{GH} which satisfies analogous properties: the (*pointed*) *Gromov-Hausdorff distance* (or *GH distance*) between two pointed metric spaces, (M,x) and (N,y), is defined by

$$d_{GH}(M,x;N,y)=\inf_d\max\{d(x,y),H_d(M,N)\}\ , \tag{21}$$

where the infimun is taken over all admissible metrics d on $M\sqcup N$.

If X is any metric space and $f:M\to X$ and $g:N\to X$ are isometric injections, then it is also well known that

$$\begin{gathered}d_{GH}(M,N)\le H_{d_X}(f(M),g(N))\ ,\\ d_{GH}(M,x;N,y)\le\max\{d_X(f(x),g(y)),H_{d_X}(f(M),g(N))\}\ ;\end{gathered} \tag{22}$$

indeed, these inequalities follow by considering, for each $\epsilon > 0$, the unique admissible metric d_ϵ on $M \sqcup N$ satisfy

$$d_\epsilon(u, v) = d_X(f(u), g(v)) + \epsilon$$

for all $u \in M$ and $v \in N$.

A metric space, or its distance function, is called *proper* (or *Heine-Borel*) if every open ball has compact closure. This condition is equivalent to the compactness of the closed balls, which means that the distance function to a fixed point is a proper function. Any proper metric space is complete and locally compact, and its cardinality is not greater than the cardinality of the continuum. Therefore it may be assumed that their underlying sets are subsets of $\mathbf{R}$. With this assumption, it makes sense to consider the set $\mathcal{M}_*$ of isometry classes, $[M, x]$, of pointed proper metric spaces, (M, x). The set $\mathcal{M}_*$ is endowed with a topology introduced by M. Gromov [4, Section 6], [3], which can be described as follows.

For a metric space X, two subspaces, $M, N \subset X$, two points, $x \in M$ and $y \in N$, and a real number $R > 0$, let $H_{d_X,R}(M, x; N, y)$ be given by

$$H_{d_X,R}(M, x; N, y) = \max\left\{ \sup_{u \in B_M(x,R)} d_X(u, N), \sup_{v \in B_N(y,R)} d_X(v, M) \right\} .$$

Then, for $R, r > 0$, let $U_{R,r} \subset \mathcal{M}_* \times \mathcal{M}_*$ denote the subset of pairs $([M, x], [N, y])$ for which there is an admissible metric, d, on $M \sqcup N$ so that

$$\max\{d(x, y), H_{d,R}(M, x; N, y)\} < r .$$

The following lemma is obtained exactly like (22).

Lemma 8.1. *For $([M, x], [N, y]) \in \mathcal{M}_* \times \mathcal{M}_*$ to be in $U_{R,r}$ it suffices that there exists a metric space, X, and isometric injections, $f : M \to X$ and $g : N \to X$, such that*

$$\max\big\{d_X(f(x), g(y)), H_{d_X,R}(f(M), f(x); g(N), g(y))\big\} < r .$$

The following notation will be used: for a relation E on $\mathcal{M}_*$ and $[M, x] \in \mathcal{M}_*$, $E([M, x])$ will be simply written as $E(M, x)$, and for a metric relation d on $\mathcal{M}_*$ and $[M, x], [N, y] \in \mathcal{M}_*$, $d([M, x], [N, y])$ will be denote by $d(M, x; N, y)$.

The sets $U_{R,r}$ obviously satisfy Hypothesis 1-(i)–(iv), and the following lemma shows that they also satisfy Hypothesis 1-(v).

Lemma 8.2. *If $R, r, s > 0$, then $U_{S,r} \circ U_{S,s} \subset U_{R,r+s}$, where $S = R + 2\max\{r, s\}$.*

Proof. Let $[M, x], [N, y] \in \mathcal{M}_*$ and $[P, z] \in U_{S,r}(N, y) \cap U_{S,s}(M, x)$. Then there are admissible metrics, d on $M \sqcup P$ and $\bar{d}$ on $N \sqcup P$, such that $d(x, z) < r$, $r_0 := H_{d,S}(M, x; P, z) < r$, $\bar{d}(y, z) < s$ and $s_0 := H_{\bar{d},S}(N, y; P, z) < s$. Let $\hat{d}$ be the admissible metric on $M \sqcup N$ such that

$$\hat{d}(u, v) = \inf\big\{ d(u, w) + \bar{d}(w, v) \mid w \in P \big\}$$

for all $u \in M$ and $v \in N$. Then

$$\hat{d}(x, y) \leq d(x, z) + \bar{d}(z, y) < r + s .$$

For each $u \in B_M(x,R)$, there is some $w \in P$ such that $d(u,w) < r_0$. Then

$$d_P(z,w) \leq d(z,x) + d_M(x,u) + d(u,w) < r + R + r_0 < S \ .$$

So there is some $v \in N$ such that $\bar{d}(w,v) < s_0$, and we have

$$\hat{d}(u,v) \leq d(u,w) + \bar{d}(w,v) < r_0 + s_0 \ .$$

Hence $\hat{d}(u,N) < r_0 + s_0$ for all $u \in B_M(x,R)$. Similarly, $\hat{d}(v,M) < r_0 + s_0$ for all $v \in B_N(y,R)$. Therefore $H_{\hat{d},R}(M,x;N,y) \leq r_0 + s_0 < r + s$. Then $[N,y] \in U_{R,r+s}(M,x)$. □

Since the sets $U_{R,r}$ satisfy Hypothesis 1, they form a base of entourages of a metrizable uniformity on $\mathcal{M}_*$. Endowed with the induced topology, $\mathcal{M}_*$ is what is called the *Gromov space* in this paper. It is well known that $\mathcal{M}_*$ is a Polish space, *e.g.* Gromov [4] or Petersen [13]; in particular, a countable dense subset is defined by the pointed finite metric spaces with **Q**-valued metrics.

Some relevant subspaces of $\mathcal{M}_*$ are defined by the following classes of metric spaces: proper ultrametric spaces, proper length spaces, connected complete Riemannian manifolds, connected locally compact simplicial complexes, connected locally compact graphs and finitely generated groups (via their Cayley graphs).

The following (generalized) dynamics can be considered on $\mathcal{M}_*$:

The canonical metric relation. The *canonical partition* E_{can} is defined by varying the distinguished point; *i.e.*, E_{can} consists of the pairs of the form $([M,x],[M,y])$ for any proper metric space M and all $x,y \in M$. There is a canonical map $M \to \mathcal{M}_*$, $x \mapsto [M,x]$, which defines an embedding $\operatorname{Isom}(M)\backslash M \to \mathcal{M}_*$ whose image is $E_{\text{can}}(M,x)$ for any $x \in M$. Observe that $\mathcal{M}_*/E_{\text{can}}$ can be identified to the set of isometry classes of proper metric spaces.

The GH metric relation. It is defined by the pointed GH distance d_{GH}. The notation $E_{GH} = E^{\mathcal{M}_*}_{d_{GH}}$ will be used. Since $E_{\text{can}} \subset E_{GH}$, the quotient set $\mathcal{M}_*/E_{GH}$ can be identified to the set of classes of proper metric spaces defined by the relation of being at finite GH distance.

The Lipschitz metric relation. The *Lipschitz partition*, E_{Lip}, is defined by the existence of pointed bi-Lipschitz bijections. It is induced by the *Lipschitz metric relation*, d_{Lip}, which is defined by using the infimum of the logarithms of the dilatations of bi-Lipschitz bijections.

The QI metric relation. The *quasi-isometric partition* (or *QI partition*), E_{QI}, is the smallest equivalence relation over $\mathcal{M}_*$ that contains $E_{GH} \cup E_{\text{Lip}}$. It is induced by the *quasi-isometric metric relation* (or *QI relation*), d_{QI}, defined as the largest metric relation over $\mathcal{M}_*$ smaller than both d_{GH} and d_{Lip} (*cf.* [14, Lemma 6]). The quotient set $\mathcal{M}_*/E_{QI}$ can be identified to the set of quasi-isometry classes of proper metric spaces.

The dilation flow. It is the multiplicative flow defined by $\lambda \cdot [M,x] = [\lambda M, x]$, where λM denotes M with its metric multiplied by λ. This flow is used to define the asymptotic and tangent cones.

The purpose of this paper is to study the GH metric relation, obtaining along the way some results about the QI metric relation.

Some technical results and concepts related to the definition of $\mathcal{M}_*$, which will be used in the next section, are given presently.

Lemma 8.3. *Let $[M,x],[N,y] \in \mathcal{M}_*$ and $r > 0$. If d is an admissible metric on $M \sqcup N$ such that $d(x,y) < r$ and $H_d(M,N) < r$, then d is proper.*

Proof. For every $v \in N$,

$$d_N(y,v) \leq d(x,y) + d(x,v) < r + d(x,v) \ ,$$

and so

$$B_d(x,R) \subset B_M(x,R) \sqcup B_N(y,R+r)$$

for all $R > 0$. The statement follows from this because M and N are proper. □

Lemma 8.4. *Let $[M,x],[N,y],[P,z] \in \mathcal{M}_*$ and $R,r > 0$. Suppose that $(B_P(z,R+2r),z)$ is isometric to $(B_N(y,R+2r),y)$, and that there is an admissible metric, d, on $M \sqcup N$ such that $d(x,y) < r$ and $H_{d,R}(M,x;N,y) < r$. Then there exists a proper admissible metric, d', on $M \sqcup P$ such that $d'(x,z) < r$ and $H_{d',R}(M,x;P,z) < r$.*

Proof. Let $A = B_M(x,R+2r)$, $B = B_N(y,R+2r)$ and $C = B_P(z,R+2r)$, and let $\phi : (B,y) \to (C,z)$ be an isometry. Let d' be the admissible metric on $M \sqcup P$ satisfying

$$d'(u,w) = \inf\{\, d_M(u,u') + d(u',v) + d_P(\phi(v),w) \mid u' \in A \ \&\ v \in B \,\}$$

for $u \in M$ and $w \in P$. Observe that $d'(u,\phi(v)) = d(u,v)$ for $u \in A$ and $v \in B$; in particular, $d'(x,z) < r$.

For each $u \in B_M(x,R)$, there is some $v \in N$ such that $d(u,v) < r$. Since

$$d_N(y,v) \leq d(y,x) + d_M(x,u) + d(u,v) < R + 2r \ ,$$

we get $d'(u,\phi(v)) = d(u,v) < r$, and therefore $d'(u,P) < r$. Similarly, $d'(w,M) < r$ for all $w \in B_P(z,R)$, obtaining $H_{d',R}(M,x;P,z) < r$.

For any $S > 0$ and $w \in P \cap B_{d'}(x,S)$, there is some $v \in B$ such that $d(x,v) + d_P(\phi(v),w) < S$. So

$$d_P(z,w) \leq d_P(z,\phi(v)) + d_P(\phi(v),w) < R + 2r + S \ ,$$

obtaining

$$B_{d'}(x,S) \subset B_M(x,S) \sqcup B_P(z,R+2r+S) \ .$$

Hence $\overline{B_{d'}(x,S)}$ is compact since M and P are proper. This shows that d' is proper. □

9 The GH metric relation

The relations $U_{R,r}$ on $\mathcal{M}_*$ defined in Section 8 satisfy Hypothesis 1 of Section 6. Consider the family of symmetric relations $E_r \subset \mathcal{M}_* \times \mathcal{M}_*$, for $r > 0$, whose fibers are $E_r(M,x) = \bigcap_{R>0} U_{R,r}(M,x)$. The notation $B_{GH}(M,x;r) = B_{d_{GH}}([M,x],r)$ will be used.

Lemma 9.1. *If $0 < r < s$, then*

$$B_{GH}(M,x;r) \subset E_r(M,x) \subset B_{GH}(M,x;s)\ .$$

Proof. The first inclusion is obvious. To prove the second one, let $[N,y] \in E_r(M,x)$. For each $R > 0$ there exists an admissible metric, d_R, on $M \sqcup N$ such that $d_R(x,y) < r$ and $H_{d_R,R}(M,x;N,y) < r$. Let ω be a free ultrafilter of $[0,\infty)$. Then there is a unique admissible metric, d, on $M \sqcup N$ such that

$$d(u,v) = \lim_{R\to\omega} d_R(u,v) + \frac{s-r}{2}$$

for all $u \in M$ and $v \in N$. For each $\epsilon > 0$ there exists $\Omega \in \omega$ such that

$$d(u,v) < d_R(u,v) + \frac{s-r}{2} + \epsilon\ ,$$

for all $R \in \Omega$. Then

$$d(x,y) \le d_R(x,y) + \frac{s-r}{2} + \epsilon < \frac{s+r}{2} + \epsilon\ ,$$

for all $R \in \Omega$, and, because this holds for each $\epsilon > 0$,

$$d(x,y) \le \frac{s+r}{2} < s\ .$$

Next, for every $u \in M$, if $R \in \Omega$ is $> d(x,u)$, then $d_R(u,N) < r$, and so $d(u,N) < s$ as before. Similarly, $d(v,M) < s$ for all $v \in N$. Therefore $H_d(M,N) < s$. $\square$

Corollary 9.2. *The metric relation over $\mathcal{M}_*$ defined by the sets $U_{R,r}$ is d_{GH}.*

Theorem 9.3. (*i*) *The metric equivalence relation (d_{GH}, E_{GH}) is turbulent;*

(*ii*) *for each Polish S_∞-space Y, the equivalence relation E_{GH} is generically $E^Y_{S_\infty}$-ergodic; and*

(*iii*) *for each Polish group G and each Polish G-space X, the equivalence relation $E_{GH} \not\le_B E^X_G$.*

According to Propositions 6.4 and 6.8–6.10, and Corollary 9.2, this theorem follows by showing that the sets $U_{R,r}$ also satisfy Hypotheses 2–4. It was already noted that $\mathcal{M}_*$ is Polish (Hypothesis 2-(i)).

Lemma 9.4. *If $R, r, s > 0$, then $U_{R+2r+s,s} \circ U_{R,r} \subset E_s \circ U_{R,r}$.*

Proof. Let $S = R+2r+s$. If $[M,x] \in \mathcal{M}_*$ and $[N,y] \in U_{S,s} \circ U_{R,r}(M,x)$, then there is $[P,z] \in U_{R,r}(M,x) \cap U_{S,s}(N,y)$. This means that there are admissible metrics, d on $M \sqcup P$ and $\bar{d}$ on $N \sqcup P$, such that $d(x,z) < r$, $H_{d,R}(M,x;P,z) < r$, $\bar{d}(y,z) < s$ and $H_{\bar{d},S}(N,y;P,z) < s$. Moreover, because of Lemma 8.4, $\bar{d}$ may be assumed to be a proper metric. The subset

$$P' = (N \setminus B_N(y,S)) \sqcup \overline{B_P(z,S)} \subset N \sqcup P$$

is closed and so it becomes a proper metric space when endowed with the metric induced by $\bar{d}$.

Claim 11. The metric space $[P',z]$ satisfies $d_{GH}(N,y;P',z) < s$.

Since $N \setminus P' \subset B_N(y,S)$ and $P' \setminus N = \overline{B_P(z,S)}$, the Hausdorff distance

$$\begin{aligned} H_{\bar{d}}(N,P') &= \max\left\{ \sup_{v \in B_N(y,S)} \bar{d}(v,P'), \sup_{w \in B_P(z,S)} \bar{d}(w,N) \right\} \\ &\le H_{\bar{d},S}(N,y;P,z) < s\ , \end{aligned}$$

and so Claim 11 follows from (22).

From Claim 11 and Corollary 9.2 it follows that $[P',z] \in E_s(N,y)$.

Claim 12. $B_{P'}(z,R+2r) = B_P(z,R+2r)$.

The inclusion "$\supset$" of this identity is obviously true. To prove that the reverse inclusion "$\subset$" is also true, it suffices to note that $B_{P'}(z,R+2r) \cap N = \emptyset$, which is true because, if there is $v \in B_{P'}(z,R+2r) \cap N$, then

$$d_N(y,v) \le \bar{d}(y,z) + \bar{d}(z,v) < s + R + 2r = S\ ,$$

which contradicts that $B_N(y,S) \cap P' = \emptyset$.

From Claim 12 and Lemma 8.4, it follows that $[P',z] \in U_{R,r}(M,x)$. Hence $[N,y] \in E_s \circ U_{R,r}(M,x)$. □

A subset A of a metric space X is called a *net* if there is an $\epsilon > 0$ such that $d_X(u,A) \le \epsilon$ for all $u \in X$, and it is called *separated* if there is some $\delta > 0$ such that $d_X(a,b) \ge \delta$ for all $a,b \in A$ with $a \ne b$; the terms ϵ-*net* and δ-*separated* are also used in these cases.

A separated subset of a metric space is discrete and therefore closed. Hence, every separated subset of a proper metric space is a proper metric space when endowed with the induced metric.

If $A \subset X$ is an ϵ-net of a metric space (X,d_X), then $H_{d_X}(X,A) \le \epsilon$. Therefore, if A is endowed with the induced metric from (X,d_X), then $d_{GH}(X,x;A,x) \le \epsilon$ for every $x \in A$ by (22); thus, by Lemma 9.1, $[A,x] \in E_\delta(X,x)$ for any $\delta > \epsilon$ if moreover X is proper and A separated.

Lemma 9.5. *Let $\epsilon > 0$. For every metric space M and every ϵ-separated subset $S \subset M$ there exists an ϵ-separated ϵ-net of M that contains S.*

Proof. By Zorn's lemma, the family of ϵ-separated subsets of M that contain S, ordered by inclusion, has a maximal element. It is easily checked that that maximal element is an ϵ-net. $\square$

The following is some kind of reverse of Lemma 8.2.

Lemma 9.6. *If $R, r, s > 0$, then $U_{R,r+s} \subset U_{R,s} \circ U_{R,r}$.*

Proof. Let $[M, x] \in \mathcal{M}_*$ and $[N, y] \in U_{R,r+s}(M, x)$. Then there is an admissible metric, d, on $M \sqcup N$ such that $d(x, y) < r_0 + s_0$ and $H_{d,R}(M, x; N, y) < r_0 + s_0$ for some $r_0 \in (0, r)$ and $s_0 \in (0, s)$. By Lemma 8.4, d may be assumed to be a proper metric.

Take any $\epsilon > 0$ such that $r_0 + 2\epsilon < r$ and $s_0 + 2\epsilon < s$. By Lemma 9.5, there are ϵ-separated ϵ-nets, A of $B_M(x, R)$ and B of $B_N(y, R)$, such that $x \in A$ and $y \in B$.

For each $u \in B_M(x, R)$, there is some $v \in N$ such that $d(u, v) < r_0 + s_0$. Then there is some $v' \in B$ so that $d_N(v, v') \leq \epsilon$. So

$$d(u, v') \leq d(u, v) + d_N(v, v') < r_0 + s_0 + \epsilon \ ,$$

giving $d(u, B) < r_0 + s_0 + \epsilon$. Similarly, $d(v, A) < r_0 + s_0 + \epsilon$ for all $v \in B_N(y, R)$.

Let Σ denote the set of pairs $(u, v) \in A \times B$ such that $d(u, v) < r_0 + s_0 + \epsilon$ and $\min\{d_M(x, u), d_N(y, v)\} < R$; in particular, $(x, y) \in \Sigma$. The set Σ is finite because A and B are separated and d is proper. For each $(u, v) \in \Sigma$, let $I_{u,v}$ denote an Euclidean segment of length $d(u, v)$, whose metric is denoted by $d_{u,v}$. Let $h : \bigsqcup_{(u,v)\in\Sigma} \partial I_{u,v} \to M \sqcup N$ be a map that restricts to a bijection $h : \partial I_{u,v} \to \{u, v\}$ for all $(u, v) \in \Sigma$. Then let

$$\widehat{P} = (M \sqcup N) \cup_h \bigsqcup_{(u,v)\in\Sigma} I_{u,v} \ .$$

The space M, N and each $I_{u,v}$ may be viewed as subspaces of $\widehat{P}$; in particular, $\partial I_{u,v} \equiv \{u, v\}$ in $\widehat{P}$. Let $\widehat{P}$ be endowed with the metric $\hat{d}$ whose restriction to $M \sqcup N$ is d, whose restriction to each $I_{u,v}$ is $d_{u,v}$, and such that

$$\begin{aligned} \hat{d}(w, w') = \min\big\{&d_{u,v}(w, u) + d_M(u, u') + d_{u',v'}(u', w'), \\ &d_{u,v}(w, v) + d_N(v, v') + d_{u',v'}(v', w')\big\} \end{aligned}$$

for $(u, v), (u', v') \in \Sigma$, $w \in I_{u,v}$ and $w' \in I_{u',v'}$.

Let $P \subset \widehat{P}$ be the finite subset consisting of the points $w \in I_{u,v}$ with $(u, v) \in \Sigma$ and

$$d_{u,v}(w, u) = \frac{r_0 + \epsilon}{r_0 + s_0 + 2\epsilon}\, d(u, v) \ .$$

Let z be the unique point in $P \cap I_{x,y}$, and consider the restriction of $\hat{d}$ to P.

If $(u, v) \in \Sigma$ and w is the unique point in $P \cap I_{u,v}$, then

$$\hat{d}(u, w) \leq d_{u,v}(u, w) < \frac{r_0 + \epsilon}{r_0 + s_0 + 2\epsilon}\, d(u, v) < r_0 + \epsilon \ .$$

So $\hat{d}(x,z) < r_0 + \epsilon < r$, $\hat{d}(u,P) < r_0 + \epsilon$ for all $u \in A$, and $\hat{d}(w,M) < r_0 + \epsilon$ for all $w \in P$. Since A is an ϵ-net in $B_M(x,R)$, it also follows that $\hat{d}(u,P) < r_0 + 2\epsilon$ for all $u \in B_M(x,R)$. Similarly, $\hat{d}(y,z) < s$, $\hat{d}(v,P) < s_0 + 2\epsilon$ for all $v \in B_N(y,R)$, and $\hat{d}(w,N) < s_0 + \epsilon$ for all $w \in P$. Thus

$$H_{\hat{d},R}(M,x;P,z) \le r_0 + 2\epsilon < r \ , \quad H_{\hat{d},R}(N,y;P,z) \le s_0 + 2\epsilon < s \ ,$$

obtaining $[P,z] \in U_{R,r}(M,x) \cap U_{R,s}(N,y)$ by Lemma 8.1. Therefore $[N,z] \in U_{R,s} \circ U_{R,r}(M,x)$. □

The following corollary gives Hypothesis 2-(ii).

Corollary 9.7. *$U_{T,r} \circ E_s \subset E_s \circ U_{R,r}$ for $R,r,s > 0$ and*

$$T = R + 2r + s + 2\max\{r,s\} \ .$$

Proof. Let $S = R + 2r + s$. By Lemmas 8.2, 9.4 and 9.6,

$$U_{T,r} \circ E_s \subset U_{T,r} \circ U_{T,s} \subset U_{S,r+s} \subset U_{S,s} \circ U_{R,r} \subset E_s \circ U_{R,r} \ . \quad \square$$

In this case, Hypothesis 2-(iii) is the statement of the next lemma.

Lemma 9.8. *For all $r,s > 0$, $[M,x] \in \mathcal{M}_*$, $[N,y] \in E_s(M,x)$, and any neighborhood V of $[N,y]$ in $\mathcal{M}_*$, there is another neighborhood W of $[N,y]$ in $\mathcal{M}_*$ such that*

$$E_r(W) \cap E_r(E_s(M,x)) \subset E_r(V \cap E_s(M,x)) \ .$$

Proof. By Lemma 8.4, there is some $S > 0$ and some open neighborhood W of $[N,y]$ in $\mathcal{M}_*$ such that, for all $[N',y'] \in \mathcal{M}_*$ and $[N'',y''] \in W$, if $(B_{N'}(y',S),y')$ is isometric to $(B_{N''}(y'',S),y'')$, then $[N',y'] \in V$. Since $[N,y] \in U_{T,s}(M,x)$ for $T = S + s + r$, we can also assume that $W \subset U_{T,s}(M,x)$.

For any $[P,z] \in E_r(W) \cap E_r(E_s(M,x))$, there are some $[N_1,y_1] \in W$ and $[N_2,y_2] \in E_s(M,x)$ such that $[P,z] \in E_r(N_1,y_1) \cap E_r(N_2,y_2)$. There are admissible metrics, d_1 on $M \sqcup N_1$ and $\bar{d}_1$ on $N_1 \sqcup P$, so that $d_1(x,y_1) < s$, $H_{d_1,T}(M,x;N_1,y_1) < s$, $\bar{d}_1(y_1,z) < r$ and $H_{\bar{d}_1,T}(N_1,y_1;P,z) < r$. Take a sequence $T_n \uparrow \infty$ in $\mathbf{R}$ with $T_0 > T$; set also $T_{-1} = T$. For each $n \in \mathbf{N}$, let $(N_{2,n},y_{2,n})$ denote an isometric copy of (N_2,y_2). Then there are admissible metrics, $d_{2,n}$ on $M \sqcup N_{2,n}$ and $\bar{d}_{2,n}$ on $N_{2,n} \sqcup P$, such that $d_{2,n}(x,y_{2,n}) < s$, $H_{d_{2,n},T_n}(M,x;N_{2,n},y_{2,n}) < s$, $\bar{d}_{2,n}(y_{2,n},z) < r$ and $H_{\bar{d}_{2,n},T_n}(N_{2,n},y_{2,n};P,z) < r$.

Let d denote the metric on $M \sqcup N_1 \sqcup (\bigsqcup_{n=0}^{\infty} N_{2,n}) \sqcup P$ which extends d_1, $\bar{d}_1$, $d_{2,n}$ and $\bar{d}_{2,n}$ for all $n \in \mathbf{N}$, and such that

$$\begin{aligned} d(u,w) = \inf\{\, & d_1(u,v_1) + \bar{d}_1(v_1,w), \\ & d_{2,n}(u,v_{2,n}) + \bar{d}_{2,n}(v_{2,n},w) \mid v_1 \in N_1,\ v_{2,n} \in N_{2,n},\ n \in \mathbf{N} \,\} \end{aligned}$$

for $u \in M$ and $w \in P$,

$$\begin{aligned} d(v_1,v_{2,n}) = \inf\big\{\, & d_1(v_1,u) + d_{2,n}(u,v_{2,n}), \\ & \bar{d}_1(v_1,w) + \bar{d}_{2,n}(w,v_{2,n}) \mid u \in M,\ w \in P \,\big\} \end{aligned}$$

for $v_1 \in N_1$ and $v_{2,n} \in N_{2,n}$, and

$$\begin{aligned} d(v_{2,m}, v_{2,n}) = \inf\{\, & d_{2,m}(v_{2,m}, u) + d_{2,n}(u, v_{2,n}), \\ & \bar{d}_{2,m}(v_{2,m}, w) + \bar{d}_{2,n}(w, v_{2,n}) \mid u \in M,\ w \in P \,\} \end{aligned}$$

for $v_{2,m} \in N_{2,m}$ and $v_{2,n} \in N_{2,n}$ with $m \neq n$. By Lemma 8.4, we can assume that the metrics d_1, $\bar{d}_1$, $d_{2,n}$ and $\bar{d}_{2,n}$ are proper for all $n \in \mathbf{N}$, and therefore d is proper as well. The set

$$N' = \overline{B_{N_1}(y_1, T)} \sqcup \left(\bigsqcup_{n=0}^{\infty} \left(\overline{B_{N_{2,n}}(y_{2,n}, T_n)} \setminus B_{N_{2,n}}(y_{2,n}, T_{n-1}) \right) \right)$$

is closed in $M \sqcup N_1 \sqcup (\bigsqcup_{n=0}^{\infty} N_{2,n}) \sqcup P$, and therefore it becomes a proper metric space with the restriction of d.

Then $H_d(M, x; N', y_1) < s$ and $H_d(N', y_1; P, z) < r$, as in Claim 11, and so $d_{GH}(M, x; N', y_1) < s$ and $d_{GH}(N', y_1; P, z) < r$ by (22), which in turn implies $[N', y_1] \in E_s(M, x) \cap E_r(P, z)$ by Lemma 9.1. On the other hand, like in Claim 12, it follows that $B_{N'}(y_1, S) = B_{N_1}(y_1, S)$, obtaining $[N', y_1] \in V$ because $[N_1, y_1] \in W$. Therefore $[P, z] \in E_r(V \cap E_s(M, x))$. □

The fact that E_{GH} has more than one class (Hypothesis 3-(i)) is obvious because any bounded metric space is at infinite GH distance from any unbounded one.

Hypothesis 3-(ii) is a consequence of the following result.

Lemma 9.9. *For all $[M, x], [N, y] \in \mathcal{M}_*$ and $R, r > 0$, there is some $s > 0$ such that $U_{R,r}(M, x) \cap E_s(N, y) \neq \emptyset$.*

Proof. Let A and B denote the balls of radius $R + 2r$ in M and N with centers x and y, respectively. For any $s_0 > d_{GH}(A, x; B, y)$, let d be an admissible metric on $A \sqcup B$ such that $d(x, y) < s_0$ and $H_d(A, B) < s_0$. Then let d' be the admissible metric on $M \sqcup N$ satisfying

$$d'(u, v) = \inf\{\, d_M(u, u') + d(u', v') + d_N(v', v) \mid u' \in A \;\&\; v' \in B \,\}$$

for all $u \in M$ and $v \in N$. Like in the proof of Lemma 8.4, it follows that d' is proper, and its restriction to $A \sqcup B$ equals d; in particular, $d'(x, y) < s_0$.

Let A' and B' denote the balls of radius $R + 2r + s_0$ in M and N with centers x and y, respectively. The set $N' = \overline{A'} \sqcup (N \setminus B')$ is closed in $M \sqcup N$, and therefore it becomes a proper metric space with the restriction of d'. Take any

$$s > \max\{s_0, R + 2r + d'(x, N \setminus B')\}\ .$$

If $N \setminus B' \neq \emptyset$, then

$$d'(u, v) \leq d_M(u, x) + d'(x, v) < R + 2r + d'(x, v)$$

for all $v \in N \setminus B'$ and $u \in \overline{A'}$, obtaining

$$H_{d'}(\overline{A'}, N \setminus B') \leq R + 2r + d'(x, N \setminus B') < s\ .$$

It follows that $H_{d'}(N, N') < s$, and therefore $d_{GH}(N, y; N', x) < s$ by (22), obtaining $[N', x] \in E_s(N, y)$ by Lemma 9.1. Like in Claim 12, we also get $B_{N'}(x, R+2r) = A$, and therefore $[N', x] \in U_{R,r}(M, x)$ by Lemma 8.4. $\square$

The proof of Hypothesis 3-(iii) is as follows. Let $R, r > 0$ and $[M, x] \in \mathcal{M}_*$, and take any $S > R$ and $s > 0$ such that $s < r$ and $R + 2\max\{s, r - s\} < S$. Let $\mathcal{D}$ denote the set of points $[N, y] \in \mathcal{M}_*$ such that there is some admissible metric, d, on $M \sqcup N$ such that $d(x, y) < s$, $H_{d,S}(M, x; N, y) < s$ and $H_d(M, N) < \infty$. Thus $\mathcal{D} \subset U_{S,s}(M, x) \cap E_{GH}(M, x)$.

Lemma 9.10. *$\mathcal{D}$ is dense in $U_{S,s}(M, x) \cap E_{GH}(M, x)$.*

Proof. It has to be shown that, for every $T, t, t' > 0$ and $[N, y] \in U_{S,s}(M, x) \cap B_{GH}(M, x; t')$, the intersection $U_{T,t}(N, y) \cap \mathcal{D} \neq \emptyset$. Let (N_1, y_1) and (N_2, y_2) be two isometric copies of (N, y). There are admissible metrics, d_1 on $M \sqcup N_1$ and d_2 on $M \sqcup N_2$, such that $d_1(x, y_1) < s$, $H_{d_1,R}(M, x; N_1, y_1) < s$, $d_2(x, y_2) < t'$ and $H_{d_2}(M, N_2) < t'$. Let $\hat{d}$ denote the metric on $M \sqcup N_1 \sqcup N_2$ whose restrictions to $M \sqcup N_1$ and $M \sqcup N_2$ are d_1 and d_2, respectively, and such that

$$\hat{d}(v_1, v_2) = \inf\{\, d_1(v_1, u) + d_2(u, v_2) \mid u \in M \,\}$$

for all $v_1 \in N_1$ and $v_2 \in N_2$. Moreover, d_2 is proper by Lemma 8.3, and d_1 can be assumed to be proper by Lemma 8.4, obtaining that $\hat{d}$ is proper as well. With

$$T' = \max\{S, T\} + 2\max\{s, t\} + t' + s \ ,$$

let $A = B_M(x, T' + 2t')$, $B_1 = B_{N_1}(y_1, T')$ and $B_2 = B_{N_2}(y_2, T')$, and define $N' = \overline{B_1} \sqcup (N_2 \setminus B_2)$. Since N' is closed in $M \sqcup N_1 \sqcup N_2$, it becomes a proper metric space with the restriction of $\hat{d}$. We have $\hat{d}(x, y_1) = d_1(x, y_1) < s$. With arguments used in Claims 11 and 12, we get $H_{\hat{d},R}(M, x; N', y_1) < s$ and

$$H_{\hat{d}}(M, N') < \max\bigl\{H_{d_1}(A, B_1), t'\bigr\} < \infty \ .$$

It follows that $[N', y_1]$ satisfies the condition to be in $\mathcal{D}$ with the restriction of $\hat{d}$ to the subset $M \sqcup N'$ of $M \sqcup N_1 \sqcup N_2$. On the other hand, since $\hat{d}(y_1, y_2) \leq t' + s$, with the arguments of Claim 12, we also get

$$B_{N'}(y_1, T + 2t) = B_{N_1}(y_1, T + 2t) \equiv B_N(y, T + 2t) \ ,$$

and therefore $[N', y_1] \in U_{T,t}(N, y)$ by Lemma 8.4. $\square$

Let $\mathcal{E}$ be the set of points $[M, x] \in \mathcal{D}$ such that M is separated (in itself). From Lemma 9.5, it easily follows that $\mathcal{E}$ is d_{GH}-dense in $\mathcal{D}$. Take any $\epsilon > 0$ such that $s + 2\epsilon < r$ and $R + 2\max\{s + \epsilon, r - s - \epsilon\} < S$. Let A be a separated ϵ-net of M that contains x, whose existence is guaranteed by Lemma 9.5, and consider the restriction of d_M to A. Observe that $[A, x] \in E_{r-s-\epsilon}(M, x)$ because $r - s - \epsilon > \epsilon$. Then the proof of Hypothesis 3-(iii) is completed by the following lemma.

Lemma 9.11. *Any point of $\mathcal{E}$ can be joined to $[A,x]$ by a d_{GH}-continuous path in $U_{R,r}(M,x)$.*

Proof. For any $[N,y] \in \mathcal{E}$, there is some admissible metric, d, on $M \sqcup N$ such that $d(x,y) < s$, $s_0 := H_{d,S}(M,x;N,y) < s$ and $s_1 := H_d(M,N) < \infty$. Moreover d is proper by Lemma 8.3. Observe that $H_{d,S}(A,x;N,y) < s_0+\epsilon$ and $H_d(A,N) \leq s_1+\epsilon$.

Let Σ be the family of pairs $(u,v) \in A \times N$ such that $d(u,v) \leq s_1 + \epsilon$ and, if $u \in B_A(x,S)$ or $v \in B_N(y,S)$, then $d(u,v) \leq s_0 + \epsilon$; in particular, $(x,y) \in \Sigma$. Like in the proof of Lemma 9.6, define $I_{u,v}$ and $d_{u,v}$ for each $(u,v) \in \Sigma$, as well as $h : \bigsqcup_{(u,v)\in\Sigma} \to A \sqcup N$,

$$\widehat{P} = (A \sqcup N) \cup_h \bigsqcup_{(u,v)\in\Sigma} I_{u,v} \ ,$$

and the metric $\hat{d}$ on $\widehat{P}$. Since d is proper and A and N are separated, the d-balls in $A \sqcup N$ are finite. Therefore, any ball in $\widehat{P}$ is contained in a finite union of segments $I_{u,v}$ and so $\widehat{P}$ is proper.

For each $t \in I = [0,1]$, let $P_t \subset \widehat{P}$ be the subset consisting of the points $w \in I_{u,v}$ with $d_{u,v}(w,u) = t\,d(u,v)$ for $(u,v) \in \Sigma$, and let z_t denote the unique point of $P_t \cap I_{x,y}$. Each P_t is a discrete subspace of $\widehat{P}$, and therefore it becomes a proper metric space with the restriction of $\hat{d}$. Moreover $(P_0,z_0) = (A,x)$ and $(P_1,z_1) = (N,y)$. For all $t,t' \in I$, $(u,v) \in \Sigma$, $w \in P_t \cap I_{u,v}$ and $w' \in P_{t'} \cap I_{u,v}$,

$$\begin{aligned} \hat{d}(w,w') &= d_{u,v}(w,w') = d(u,v)\,|t-t'| \\ &\leq \begin{cases} (s_1+\epsilon)\,|t-t'| & \text{for arbitrary } (u,v) \in \Sigma \\ (s_0+\epsilon)\,|t-t'| & \text{if } u \in B_A(x,S) \text{ or } v \in B_N(y,S) \ . \end{cases} \end{aligned} \tag{23}$$

Thus $\hat{d}(z_t,z_{t'}) \leq (s_0+\epsilon)\,|t-t'|$ and $H_{\hat{d}}(P_t,P_{t'}) \leq (s_1+\epsilon)\,|t-t'|$. By (22), it follows that $[P_t,z_t] \in E_{GH}(M,x)$ for all $t \in I$, and the mapping $t \mapsto [P_t,z_t]$ is d_{GH}-continuous.

From (23), it also follows that $\hat{d}(u,P_t) \leq (s_0+\epsilon)t$ for all $u \in B_A(x,S)$ and $t \in I$. Moreover the ball $B_{P_t}(z_t,S)$ is contained in the union of the segments $I_{u,v}$ for $(u,v) \in \Sigma$ with $u \in B_A(x,S)$ or $v \in B_N(y,S)$. So $\hat{d}(w,P_t) \leq (s_0+\epsilon)t$ for all $w \in B_P(z_t,S)$ by (23). It follows that

$$H_{\hat{d},S}(A,x;P_t,z_t) \leq (s_0+\epsilon)t < s+\epsilon \ ,$$

obtaining

$$[P_t,z_t] \in U_{S,s+\epsilon}(A,x) \subset U_{S,s+\epsilon} \circ E_{r-s-\epsilon}(M,x) \subset U_{R,r}(M,x)$$

by Lemmas 8.1 and 8.2. $\square$

Hypotheses 1-3 have just been proved, and that suffices to obtain Theorem 9.3-(i)–(iii). In particular, d_{GH} is turbulent, and therefore the equivalence classes of E_{GH} are meager in $\mathcal{M}_*$. Then Hypothesis 4 follows from the following lemma because the compact metric spaces define one class of E_{GH}.

Lemma 9.12. *If $0 < r < s$, then $E_s(N,y) \setminus E_r(M,x)$ is dense in $E_s(N,y)$ for all $[N,y]$ and $[M,x]$ in $\mathcal{M}_*$ so that N is unbounded.*

Proof. It has to be shown that, for every $T, t > 0$ and $[P,z] \in E_s(N,y)$, the intersection $U_{T,t}(P,z) \cap (E_s(N,y) \setminus E_r(M,x)) \neq \emptyset$. Let d be an admissible metric on $N \sqcup P$ such that $d(y,z) < s$ and $H_{d,T+2t+s}(N,y;P,z) < s$. Take $s_0, \epsilon, \delta > 0$ such that

$$\max\{r, H_{d,T+2t+s}(N,y;P,z)\} < s_0 < s\ , \quad \epsilon < \delta\ , \quad r + 2\delta < s_0\ .$$

By Lemma 9.5, M has some separated ϵ-net, A, which contains x. Then $[A,x] \in E_\delta(M,x)$.

Take a point $v_0 \in N \setminus \overline{B_N(y, T+2t+s)}$ (this is possible because N is unbounded). Then, for $S = d_N(y,v_0)$, the cardinality $n := \#B_A(x, S+3s) < \infty$ because M is proper and A separated in M.

Let $w_0, \dots, w_n$ be $n+1$ different points, and let $\hat{d}$ be the metric on $P \sqcup N \sqcup \{w_0, \dots, w_n\}$ whose restriction to $N \sqcup P$ is d, such that $\hat{d}(w_i, w_j) = 2s_0$ if $i \neq j$, and $\hat{d}(v, w_i) = d(v, v_0) + s_0$ for any $v \in N \sqcup P$. It can be assumed that d is proper by Lemma 8.4, and therefore $\hat{d}$ is proper as well. The set

$$Q = \overline{B_P(z, T+2t+s)} \sqcup (N \setminus B_N(x, T+2t+s)) \sqcup \{w_0, \dots, w_n\}$$

is closed in $P \sqcup N \sqcup \{w_0, \dots, w_n\}$. So Q becomes a proper metric space with the restriction of $\hat{d}$. Then $\hat{d}(y,z) = d(y,z) < s$, and, as in Claim 11, $H_{\hat{d}}(N,Q) \leq s_0$. Then $d_{GH}(N,y;Q,z) < s$ by (22). Hence $[Q,z] \in E_s(N,y)$ by Lemma 9.1.

As in Claim 12, the balls $B_Q(z, T+2t) = B_P(z, T+2t)$, and so $[Q,z] \in U_{T,t}(N,y)$ by Lemma 8.4.

If $[Q,z] \in E_r(M,x)$, then $[Q,z] \in E_{r+\delta}(A,x)$ by (14). Therefore there is an admissible metric, $\bar{d}$, on $A \sqcup Q$ such that $\bar{d}(x,z) \leq r + \delta$ and $H_{\bar{d},S+s}(A,x;Q,z) \leq r + \delta$. For each $i \in \{0, \dots, n\}$,

$$d_Q(z, w_i) \leq \hat{d}(z,y) + d_N(y, w_i) < s + S\ ,$$

and therefore there is some $u_i \in A$ with $\bar{d}(u_i, w_i) < r + \delta$. Hence

$$2s_0 = d_Q(w_i, w_j) \leq \bar{d}(w_i, u_i) + \bar{d}(u_i, w_j) < r + \delta + \bar{d}(u_i, w_j)$$

for $j \neq i$, which gives

$$\bar{d}(u_i, w_j) > 2s_0 - r - \delta > r + \delta\ .$$

It follows that $u_i \neq u_j$ if $i \neq j$. But

$$d_M(u_i, x) \leq \bar{d}(u_i, w_i) + d_Q(w_i, z) + \bar{d}(z, x) < S + 3s\ ,$$

obtaining the contradiction $\#B_A(x, S+3s) \geq n+1$. So $[Q,z] \notin E_r(M,x)$. □

Remark 16. Like in Remark 15, it can be proved that $\emptyset$ is residual in $B_{GH}(M,x;r)$ for all $r > 0$ if M is unbounded. In this case, for sequences $0 < r_n \uparrow r$ and $0 < R_n \uparrow \infty$, consider the sets U_n consisting of the points $[N,y] \in B_{GH}(M,x;r)$ such that

$$H_d\left(M \setminus \overline{B_M(x,R_n)}, N \setminus \overline{B_N(y,R_n)}\right) > r_n$$

for every admissible metric, d, on $M \sqcup N$.

10 The QI metric relation

Consider the notation of Sections 8 and 9.

Proposition 10.1. *The fibers of E_{QI} are meager in $\mathcal{M}_*$.*

The proof of Proposition 10.1 requires an analysis of d_{QI}, which in turn requires an analysis of d_{GH} and d_{Lip}. As noted in Section 8, d_{Lip} is the metric equivalence relation over $\mathcal{M}_*$ defined by setting $d_{\mathrm{Lip}}(M, x; N, y)$ equal to the infimum of all $r \geq 0$ such that there is a pointed bi-Lipschitz bijection $\phi : (M, x) \to (N, y)$ with

$$e^{-r} d_M(u, v) \leq d_N(\phi(u), \phi(v)) \leq e^r d_M(u, v) \tag{24}$$

for all $u, v \in M$; in particular, $d_{\mathrm{Lip}}(M, x; N, y) = \infty$ if there is no such a ϕ. On the other hand, $d_{QI}(M, x; N, y)$ equals the infimum of all sums

$$\sum_{i=1}^{k} d_{GH}(M_{2i-2}, x_{2i-2}; M_{2i-1}, x_{2i-1}) + d_{\mathrm{Lip}}(M_{2i-1}, x_{2i-1}; M_{2i}, x_{2i})$$

for finite sequences $[M, x] = [M_0, x_0], \ldots, [M_{2k}, x_{2k}] = [N, y]$ in $\mathcal{M}_*$, where k is an arbitrary positive integer. For $[M, x] \in \mathcal{M}_*$ and $r > 0$, the notation $B_{\mathrm{Lip}}(M, x; r) = B_{d_{\mathrm{Lip}}}([M, x], r)$ and $B_{QI}(M, x; r) = B_{d_{QI}}([M, x], r)$ will be used.

Lemma 10.2. *If $[N, y] \in U_{R,r}(M, x)$ and $B_M(x, q) \setminus \overline{B_M(x, p)} \neq \emptyset$ for $r > 0$ and $R \geq q > p > 2r$, then $B_N(y, q + 2r) \setminus \overline{B_N(y, p - 2r)} \neq \emptyset$.*

Proof. By hypothesis, there is some admissible metric d on $M \sqcup N$ so that $d(x, y) < r$ and $H_{d,R}(M, x; N, y) < r$, and there is some $u \in M$ such that $p < d(x, u) < q$. Since $H_{d,R}(M, x; N, y) < r$, there is some $v \in N$ so that $d(u, v) < r$. Then

$$d_N(y, v) \leq d(x, u) + d(y, x) + d(u, v) < q + 2r \;,$$
$$d_N(y, v) \geq d(x, u) - d(y, x) - d(u, v) > p - 2r \;,$$

completing the proof. □

Corollary 10.3. *If $d_{GH}(M, x; N, y) < r$ and $B_M(x, q) \setminus \overline{B_M(x, p)} \neq \emptyset$ for $r > 0$ and $q > p > 2r$, then $B_N(y, q + 2r) \setminus \overline{B_N(y, p - 2r)} \neq \emptyset$.*

Lemma 10.4. *If $d_{\mathrm{Lip}}(M, x; N, y) < r$ and $B_M(x, q) \setminus \overline{B_M(x, p)} \neq \emptyset$ for some $r > 0$ and $q > p > 0$, then $B_N(y, e^r q) \setminus \overline{B_N(y, e^{-r} p)} \neq \emptyset$.*

Proof. By hypothesis, there is some pointed bi-Lipschitz bijection $\phi : (M, x) \to (N, y)$ satisfying (24), and there is some $u \in M$ such that $p < d(x, u) < q$. Then

$$d_N(y, \phi(u)) \leq e^r d_M(x, u) < e^r q \;,$$
$$d_N(y, \phi(u)) \geq e^{-r} d_M(x, u) > e^{-r} p \;,$$

showing the result. □

Proof of Proposition 10.1. Recall that the pointed compact spaces define a class of E_{GH}, which is meager in $\mathcal{M}_*$ by Theorem 9.3-(i). Moreover any metric space bi-Lipschitz equivalent to a bounded one is also bounded. So the pointed compact metric spaces also form a class of E_{QI}. Thus, to prove Proposition 10.1, it is enough to consider the fiber $E_{QI}(M, y)$ for any unbounded proper metric space M. Hence there are sequences $p_n, q_n \uparrow \infty$ such that $q_n > p_n > 0$ and $B_M(x, q_n) \setminus \overline{B_M(x, p_n)} \neq \emptyset$.

Claim 13. Let $r, s > 0$ and $n \in \mathbf{N}$ so that $p_n > 2r$ and $2s < e^{-r}(q_n - 2r)$. If $[N, y] \in \overline{B_{QI}(M, x; r)}$, then

$$B_N(y, e^r(q_n + 2r) + 2s) \setminus \overline{B_N(y, e^{-r}(p_n - 2r) - 2s)} \neq \emptyset \ . \tag{25}$$

To prove this assertion, fix any $S > e^r(q_n + 2r)$. Since $[N, y] \in \overline{B_{QI}(M, x; r)}$, there is a finite sequence, $[M, x] = [M_0, x_0], \dots, [M_{2k}, x_{2k}]$ in $\mathcal{M}_*$, for some positive integer k, such that $[M_{2k}, x_{2k}] \in U_{S,s}(N, y)$ and

$$\sum_{i=1}^{k} d_{GH}(M_{2i-2}, x_{2i-2}; M_{2i-1}, x_{2i-1}) + d_{\text{Lip}}(M_{2i-1}, x_{2i-1}; M_{2i}, x_{2i}) < r \ .$$

Take $r_1, \dots, r_{2k} > 0$ such that $\sum_{j=1}^{2k} r_j < r$ and

$$r_j > \begin{cases} d_{GH}(M_{j-1}, x_{j-1}; M_j, x_j) & \text{if } j \text{ is odd} \\ d_{\text{Lip}}(M_{j-1}, x_{j-1}; M_j, x_j) & \text{if } j \text{ is even} \end{cases}$$

for $j \in \{1, \dots, 2k\}$. Let $\bar{r}_j = \sum_{a=1}^{j} r_a$. Arguing by induction on j, using Corollary 10.3 and Lemma 10.4, it follows that

$$B_{M_j}(x_j, e^{\bar{r}_j}(q_n + 2\bar{r}_j)) \setminus \overline{B_{M_{2k}}(x_{2k}, e^{-\bar{r}_j}(q_n - 2\bar{r}_j))} \neq \emptyset$$

for all j. So

$$B_{M_{2k}}(x_{2k}, e^r(q_n + 2r)) \setminus \overline{B_{M_{2k}}(x_{2k}, e^{-r}(q_n - 2r))} \neq \emptyset \ .$$

Then (25) follows by Lemma 10.2, completing the proof of Claim 13.

Since $E_{QI}(M, x) = \bigcup_{r=1}^{\infty} B_{QI}(M, x; r)$, the result follows from the following claim.

Claim 14. $B_{QI}(M, x; r)$ is nowhere dense in $\mathcal{M}_*$ for each $r > 0$.

Let $[N, y] \in \overline{B_{QI}(M, x; r)}$. Given $S, s > 0$, there is some $n \in \mathbf{N}$ such that $p_n > 2r$ and $S < e^{-r}(q_n - 2r) - 2s$. Thus (25) is satisfied with these $[N, y]$, r, s and n. Let

$$N' = N \setminus \left(B_N(y, e^r(q_n + 2r) + 2s) \setminus \overline{B_N(y, e^{-r}(q_n - 2r) - 2s)} \right) \ .$$

Endowed with the restriction of d_N, N' becomes a proper metric space with $B_{N'}(y, S) = B_N(y, S)$, obtaining $[N', y] \in U_{S,s}$. But $[N', y] \notin \overline{B_{QI}(M, x; r)}$ by Claim 13 because

$$B_{N'}(y, e^r(q_n + 2r) + 2s) \setminus \overline{B_{N'}(y, e^{-r}(p_n - 2r) - 2s)} = \emptyset \ .$$

So $U_{S,s}(N, y) \not\subset \overline{B_{QI}(M, x; r)}$. Then Claim 14 follows since s can be chosen arbitrarily small, and S can be chosen arbitrarily large by chosing n arbitrarily large. $\square$

Theorem 10.5. (i) *The metric equivalence relation* (d_{QI}, E_{QI}) *is turbulent.*

(ii) *For each Polish* S_∞*-space* Y*, the equivalence relation* E_{QI} *is generically* $E^Y_{S_\infty}$*-ergodic.*

Proof. By Theorem 9.3-(i) and Remarks 6 and 8, E_{QI} is minimal and the local equivalence classes of d_{QI} are somewhere dense. Moreover the equivalence classes of E_{QI} are meager by Proposition 10.1. So (i) holds.

Property (ii) is a direct consequence of Theorem 9.3-(ii) and Remark 3. □

References

[1] Howard Becker and Alexander S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996. MR 1425877 (98d:54068)

[2] Su Gao and Alexander S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766, viii+78. MR 1950332 (2004b:03067)

[3] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR 623534 (83b:53041)

[4] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999, Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR 1699320 (2000d:53065)

[5] Greg Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, Providence, RI, 2000. MR 1725642 (2000k:03097)

[6] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[7] ______, *Descriptive dynamics*, Descriptive set theory and dynamical systems (Marseille-Luminy, 1996), London Math. Soc. Lecture Note Ser., vol. 277, Cambridge Univ. Press, Cambridge, 2000, pp. 231–258. MR 1774428 (2001j:37038)

[8] Alexander S. Kechris and Benjamin D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag, Berlin, 2004. MR 2095154 (2005f:37010)

[9] Casimir Kuratowski, *Topologie. Vol. I*, Monografie Matematyczne, Tom 20, Państwowe Wydawnictwo Naukowe, Warsaw, 1958, 4ème éd. MR 0090795 (19,873d)

[10] Byron L. McAllister, *Hyperspaces and multifunctions, the first half century (1900–1950)*, Nieuw Arch. Wisk. (3) **26** (1978), no. 2, 309–329. MR 501288 (80g:54001)

[11] Ernest Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182. MR 0042109 (13,54f)

[12] ______, *Continuous selections. I*, Ann. of Math. (2) **63** (1956), 361–382. MR 0077107 (17,990e)

[13] Peter Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998. MR 1480173 (98m:53001)

[14] Frank Plastria, *Asymmetric distances, semidirected networks and majority in Fermat-Weber problems*, Ann. Oper. Res. **167** (2009), 121–155. MR 2506484 (2010f:90078)

[15] Simon Thomas, *On the complexity of the quasi-isometry and virtual isomorphism problems for finitely generated groups*, Groups Geom. Dyn. **2** (2008), no. 2, 281–307. MR 2393184 (2009b:03134)

[16] A. M. Vershik, *The universal Uryson space, Gromov's metric triples, and random metrics on the series of natural numbers*, Uspekhi Mat. Nauk **53** (1998), no. 5(323), 57–64. MR 1691182 (2000b:53055)

[17] Leopold Vietoris, *Bereiche zweiter Ordnung*, Monatsh. Math. Phys. **32** (1922), no. 1, 258–280. MR 1549179